1. Problems from Ross.

2. (29.2) Recall that if $f(x) = \cos x$, then $f'(x) = -\sin x$. Ergo since the cosine function is continuous and differentiable everywhere on $\mathbb{R}$, for any $y < x \in \mathbb{R}$, by the Mean Value Theorem there is some $z \in (y, x)$ such that

$$\frac{\cos x - \cos y}{x - y} = f'(z) = -\sin z$$

Taking the absolute value of both sides and multiplying by $|x - y|$, we see that $|\cos x - \cos y| = |\sin z||x - y|$. However, $|\sin z| \leq 1$ for any $z$, so in fact $|\cos x - \cos y| \leq |x - y|$.

3. (29.3) (a) First, observe that since $f$ is differentiable on $\mathbb{R}$, we have $f$ continuous on $\mathbb{R}$. Ergo $f$ is in particular continuous on $[0, 2]$ and differentiable on $(0, 2)$, so by MVT, we see that there is some $x$ in $(0, 2)$ such that $f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{1}{2}$.

(b) First observe that $f$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$, so by MVT, there is some $y \in (1, 2)$ with the property that $f'(y) = \frac{f(2) - f(1)}{2 - 1} = 0$. But this means there is some $x$ in $(0, 2)$ with $f'(x) = \frac{1}{2}$ from part (a) and some $y \in (0, 2)$ with $f'(y) = 0$. Since derivatives of functions over intervals have the intermediate value property, we see there is some $z$ between $x$ and $y$ with $f'(z) = \frac{1}{2}$.

4. (29.4) Let $f$ and $g$ be differentiable functions on an interval $I$, and let $f(a) = f(b) = 0$ for some $a < b$ on $I$. Following the hint, we consider the function $h(x) = f(x)e^{g(x)}$. Then $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $h(a) = 0 = h(b)$. Moreover, we see that in general $h'(x) = f'(x)e^{g(x)} + f(x)g'(x)e^{g(x)}$. Therefore by the Mean Value Theorem, there is some $z \in (a, b)$ such that

$$0 = \frac{h(b) - h(a)}{a - b} = f'(z)e^{g(z)} + f(z)g'(z)e^{g(z)}$$

Ergo $0 = (f'(x) + f(x)g'(x))e^{g(x)}$. But since $e^{g(x)} \neq 0$, in fact we have $0 = f'(x) + f(x)g'(x)$.

(29.5) Suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y$ in $\mathbb{R}$. Then for any given $a \in \mathbb{R}$, we have $|\frac{f(x) - f(a)}{x - a}| \leq |x - a|$, so by the squeeze theorem, as $x \to a$, $|\frac{f(x) - f(a)}{x - a}| \to 0$. Therefore $f'(a) = 0$. Since $a$ was arbitrary, $f'(x)$ is identically zero on $\mathbb{R}$, so by Corollary
29.4 \( f \) must be a constant function.

(29.9) Let \( f(x) = e^x - ex \). We claim this function is nonnegative everywhere. First, observe that \( f(1) = 0 \). Now observe that \( f'(x) = e^x - e \) is positive on \((1, \infty)\) and negative on \((-\infty, 1)\), so \( f \) is decreasing on \((-\infty, 1)\) and increasing on \([1, \infty)\). In particular \( f(1) = 0 \) is the minimum value taken by \( f \). Ergo \( e^x \geq ex \) everywhere.

(29.13) Suppose \( f, g \) are differentiable, \( f(0) = g(0) \), and \( f'(x) \leq g'(x) \) on \( \mathbb{R} \). Consider the function \( g - f(x) \). Observe that \( g - f(0) = 0 \), and 
\[
(g - f)'(x) = g'(x) - f'(x) \geq 0 \quad \text{for} \quad x \geq 0.
\]
Therefore \( g - f \) is an increasing function, so by Corollary 29.7 \( g(x) \geq f(x) \).

(29.16) Let \( x = g(y) = \arctan y \) be the inverse of \( y = f(x) = \tan x \) on \((-\pi/2, \pi/2)\). By an application of the quotient rule, we know \( f'(x) = \sec^2(x) \neq 0 \) on \((-\pi/2, \pi/2)\). Therefore by Theorem 29.9, 
\[
g'(y) = \frac{1}{\sec^2(x)} = \frac{1}{1+y^2}.
\]
This is especially interesting because \( \arctan \) is a transcendental function whose derivative is an algebraic function! So apparently the line between these categories is not as distinct as it might seem.

(29.18) Let \( f \) be differentiable on \( \mathbb{R} \) with \( a = \sup\{|f'(x)| : x \in \mathbb{R}\} \).
(a) Choose \( s_0 \), and recursively define \( s_n = f(s_{n-1}) \) for \( n \geq 1 \). Observe that 
\[
|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| \leq a|s_n - s_{n-1}|.
\]
Therefore by the mean value theorem, there is some \( y_n \) between \( s_{n-1} \) and \( s_n \) such that
\[
\left| \frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} \right| = |f'(y_n)| < a.
\]
In particular, \( |s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| < a|s_n - s_{n-1}| < a^{n-1}|s_1 - s_0| \). We claim this implies \( s_n \) is Cauchy. For given \( \epsilon > 0 \), there exists some \( N \) such that
\[ \frac{a^N}{1-a}|s_1 - s_0| < \epsilon. \] Then for \( n > m > N \), we have

\[
|s_n - s_m| \leq \sum_{k=m+1}^{n} |s_k - s_{k+1}|
\]
\[
\leq \sum_{k=m+1}^{n} a^{k-1}|s_1 - s_0|
\]
\[
= (a^m + a^{m+1} + \cdots + a^{n-1})|s_1 - s_0|
\]
\[
= a^m(1 + a + a^2 + \cdots + a^{n-m-1})|s_1 - s_0|
\]
\[
\leq \frac{a^m}{1-a}|s_1 - s_0|
\]
\[
\leq \frac{a^N}{1-a}|s_1 - s_0|
\]
\[
< \epsilon.
\]

Ergo \((s_n)\) is Cauchy, hence converges to some \(s\).

(b) Notice that \((f(s_n)) = (s_{n+1})\), so if \(s_n \rightarrow s\), \(f(s_n) \rightarrow s\) as well. By continuity, \(f(s) = s\), and \(s\) is a fixed point of \(f\).