Homework 12 Solutions

MTH 320

1. Problems from Ross.

2. (29.2) Recall that if $f(x) = \cos x$, then $f'(x) = -\sin x$. Ergo since the cosine function is continuous and differentiable everywhere on $\mathbb{R}$, for any $y < x \in \mathbb{R}$, by the Mean Value Theorem there is some $z \in (y, x)$ such that

$$\frac{\cos x - \cos y}{x - y} = f'(z) = -\sin z$$

   Taking the absolute value of both sides and multiplying by $|x - y|$, we see that $|\cos x - \cos y| = |\sin z||x - y|$. However, $|\sin z| \leq 1$ for any z, so in fact $|\cos x - \cos y| \leq |x - y|$.

3. (29.3) (a) First, observe that since f differentiable on $\mathbb{R}$, we have f continuous on $\mathbb{R}$. Ergo f is in particular continuous on $[0, 2]$ and differentiable on $(0, 2)$, so by MVT, we see that there is some x in $(0, 2)$ such that $f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{1}{2}$.

   (b) First observe that f is continuous on $[1, 2]$ and differentiable on $(1, 2)$, so by MVT there is some $y \in (1, 2)$ with the property that $f'(y) = \frac{f(2)-f(1)}{2-1} = 0$. But this means there is some x in $(0, 2)$ with $f'(x) = \frac{1}{2}$ from part (a) and some $y \in (0, 2)$ with $f'(y) = 0$. Since derivatives of functions over intervals have the intermediate value property, we see there is some z between x and y with $f'(z) = \frac{1}{7}$.

4. (29.4) Let f and g be differentiable functions on an interval I, and let $f(a) = f(b) = 0$ for some $a < b$ on I. Following the hint, we consider the function $h(x) = f(x)e^{g(x)}$. Then h is continuous on $[a, b]$ and differentiable on (a, b), and $h(a) = 0 = h(b)$. Moreover, we see that in general $h'(x) = f'(x)e^{g(x)} + f(x)g'(x)e^{g(x)}$. Therefore by the Mean Value Theorem, there is some $z \in (a, b)$ such that

$$0 = \frac{h(b) - h(a)}{a - b} = f'(z)e^{g(z)} + f(z)g'(z)e^{g(z)}$$

   Ergo $0 = (f'(x) + f(x)g'(x))e^{g(x)}$. But since $e^{g(x)} \neq 0$, in fact we have $0 = f'(x) + f(x)g'(x)$.

   (29.5) Suppose $|f(x) - f(y)| \leq (x - y)^2$ for all x, y in $\mathbb{R}$. Then for any given $a \in \mathbb{R}$, we have $|\frac{f(x)-f(a)}{x-a}| \leq |x - a|$, so by the squeeze theorem, as $x \to a$, $|\frac{f(x)-f(a)}{x-a}| \to 0$. Therefore $f'(a) = 0$. Since a was arbitrary, $f'(x)$ is identically zero on $\mathbb{R}$, so by Corollary

29.4 f must be a constant function.

(29.9) Let $f(x) = e^x - ex$. We claim this function is nonnegative everywhere. First, observe that $f(1) = 0$. Now observe that $f'(x) = e^x - e$ is positive on $(1, \infty)$ and negative on $(-\infty, 1)$, so f is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$. In particular $f(1) = 0$ is the minimum value taken by f. Ergo $e^x \geq ex$ everywhere.

(29.13) Suppose f, g are differentiable, $f(0) = g(0)$, and $f'(x) \leq g'(x)$ on $\mathbb{R}$. Consider the function $g - f(x)$. Observe that $g - f(0) = 0$, and $(g - f)'(x) = g'(x) - f'(x) \geq 0$ for $x \geq 0$. Therefore $g - f$ is an increasing function, so by Corollary 29.7 $x \geq 0$, $g - f(x) \geq g - f(0) = 0$, or equivalently for $x \geq 0$, $g(x) \geq f(x)$.

(29.16)Let $x = g(y) = \arctan y$ be the inverse of $y = f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. By an application of the quotient rule, we know $f'(x) = \sec^2(x) \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore by Theorem 29.9, $g'(y) = \frac{1}{\sec^2(x)}$. It remains to turn this into an expression in y. Recall that $y = \tan x$, so drawing an appropriate right triangle shows that $\sec x = \frac{1}{\sqrt{y^2+1}}$. Therefore $g'(x) = \frac{1}{1+y^2}$.

This is especially interesting because arctan is a transcendental function whose derivative is an algebraic function! So apparently the line between these categories is not as distinct as it might seem.

(29.18) Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\}$.

(a) Choose s_0, and recursively define $s_n = f(s_{n-1})$ for $n \geq 1$. Observe that $|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})|$. Therefore by the mean value theorem, there is some y_n between s_{n-1} and s_n such that

$$\left| \frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} \right| = |f'(y_n)| < a$$

In particular, $|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| < a|s_n - s_{n-1}| < a^{n-1}|s_1 - s_0|$. We claim this implies s_n is Cauchy. For given $\epsilon > 0$, there exists some N such that

$\frac{a^N}{1-a}|s_1 - s_0| < \epsilon$. Then for $n > m > N$, we have

$$
\begin{aligned}
|s_n - s_m| &\le \sum_{k=m+1}^{n} |s_k - s_{k+1}| \\
&\le \sum_{k=m+1}^{n} a^{k-1}|s_1 - s_0| \\
&= (a^m + a^{m+1} + \cdots + a^{n-1})|s_1 - s_0| \\
&= a^m(1 + a + a^2 + \cdots + a^{n-m-1})|s_1 - s_0| \\
&\le \frac{a^m}{1-a}|s_1 - s_0| \\
&\le \frac{a^N}{1-a}|s_1 - s_0| \\
&< \epsilon.
\end{aligned}
$$

Ergo (s_n) is Cauchy, hence converges to some s.

(b) Notice that $(f(s_n)) = (s_{n+1})$, so if $s_n \to s$, $f(s_n) \to s$ as well. By continuity, $f(s) = s$, and s is a fixed point of f.