Homework 11 Solutions

MTH 320

- (25.3)(a) Clearly the pointwise limit of (f_n) is the constant function $f(x) \equiv \frac{1}{2}$. We have
$$|f_n(x) - f(x)| = \left|\frac{\cos x}{2n + \sin^2 x}\right| \leq \frac{1}{2n}.$$
So for $\epsilon > 0$, if we choose N such that $\frac{1}{2N} < \epsilon$, for $n > N$ and any $x \in \mathbb{R}$ we have $|f_n(x) - f(x)| \leq \frac{1}{2n} < \epsilon$.

  (b) The limit is the integral of the limit function, to wit, $\int_2^7 \frac{1}{2} dx = \frac{5}{2}$.

- (25.6) (a) If $\sum |a_k|$ converges, then observe that for $x \in [-1, 1]$, we have $|a_k x^k| < M_k = |a_k|$. So by the M-test, we see that $\sum a_k x^k$ converges uniformly on $[-1, 1]$. Since each term $a_k x^k$ is continuous, the limit function must itself be continuous.

  (b)Yes, by part (a), since $\sum \frac{1}{n^2}$ converges.

- (25.7) Observe that for all $x \in \mathbb{R}$, $|\frac{1}{n^2}\cos(nx)| < \frac{1}{n^2} = M_n$. Since $\sum \frac{1}{n^2}$ converges, by the Weierstrass M-test $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)$ converges uniformly on $\mathbb{R}$. In particular since each term of the series is continuous, the limit function must be continuous.

- (25.10) (a) If $x = 0$ every term of the series is 0 and it converges. If $0 < x < 1$, we see that $|\frac{a_{n+1}}{a_n}| = x \cdot \frac{1+x^n}{1+x^{n+1}} \to x < 1$, so by the Ratio Test the series converges at x.

  (b) For $x \in [0, a]$, $a < 1$, we see that $\frac{x^n}{1+x^n} < a^n$. But $\sum a^n$ converges, so by the M-test, $\sum \frac{x^n}{1+x^n}$ converges uniformly on $[0, a]$.

  (c) No. Notice that $\frac{x^n}{1+x^n} > \frac{x^n}{2}$, and $\sum_{n=0}^{\infty} \frac{x^n}{2} = \frac{1}{2(1-x)}$ is unbounded on $[0, 1)$. So $\sum \frac{x^n}{1+x^n}$ is unbounded on $[0, 1)$. But each partial sum of the series is bounded on $[0, 1)$ (for example because it is a continous function on $[0, 1]$, and therefore bounded by the Extreme Value Theorem). The uniform limit of bounded functions is necessarily bounded.

- (28.2)(a) We compute that $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} (x-2)(x^2+2x+4)x - 2 = \lim_{x\to 2}(x^2 + 2x + 4) = 12$.

  (b) We compute that $\lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{x+2-(a+2)}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = 1$.

(c) We compute that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \cos x}{x} = \lim_{x\to 0} x\cos x = 0\cos 0 = 0$.

(d) We compute that $\lim_{x\to 1} \frac{r(x)-r(1)}{x-1} = \lim_{x\to 1} \frac{\frac{3x+4}{2x-1}-7}{x-1} = \lim_{x\to 1} \frac{-11x+11}{(x-1)(2x-1)} = \lim_{x\to 1} \frac{-11}{(2x-1)} = -11$.

- (28.5) (a) Yes, f and g are differentiable on all of $\mathbb{R}$. The only point that we need to check carefully is $f(x)$ at $x=0$. We compute

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x} \\ &= \lim_{x\to 0} x \sin \frac{1}{x} \\ &= 0\end{aligned}$$

  Here the last step follows from the squeeze theorem for functions, since $-x \le x\sin(\frac{1}{x}) \le x$, and $\lim_{x\to 0} -x = 0 = \lim_{x\to 0} -x$. Therefore $f'(0) = 0$, and f is differentiable everywhere as promised.

  (b) If $x = \frac{1}{\pi n}$, $f(x) = (\frac{1}{\pi n})^2 \sin(\pi n) = 0$.

  (c)The expression $\lim_{x\to 0} \frac{g(f(x))-g(f(0))}{f(x)-f(0)}$ is meaningless because there are points x arbitrarily close to 0 for which $f(x) = 0 = f(0)$, and hence for which this expression is undefined.

- (28.8) (a) Let x_n be any sequence of points in $\mathbb{R}$ with $x_n \to 0$. Then if x_n is rational, $f(x_n) = x_n^2$, and when x_n is irrational, $f(x_n) = 0$, so in either case, $0 \le f(x_n) \le x_n^2$. Ergo since $\lim_{n\to\infty} x_n^2 = 0$, by the squeeze theorem for sequences, $\lim f(x_n) = 0 = f(0)$. Since (x_n) was arbitrary, f is continuous at 0.

  (b) Let $x \ne 0$. Then we may pick a sequence of rational numbers (x_n) such that $x_n \to x$, and also a sequence of irrational numbers (y_n) such that $y_n \to x$. We observe that $\lim f(x_n) = \lim x_n^2 = x^2$, but $\lim f(y_n) = \lim 0 = 0$. Since $x^2 \ne 0$, $\lim f(x_n) \ne \lim f(y_n)$, so f cannot be continuous at x.

  (c) Observe that for $x \ne 0$ the quotient $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$ is equal to 0 when x is irrational and is equal to x when x is rational. So for all x, we see that $|\frac{f(x)-f(0)}{x-0}| < |x|$, or alternately $-|x| \le \frac{f(x)-f(0)}{x-0} \le |x|$. Therefore by the squeeze theorem for functions, since $\lim_{x\to 0} |x| = 0$, $\lim \frac{f(x)-f(0)}{x-0} = 0 = f'(0)$.

- (28.14) Let f be differentiable at a, i.e, let there the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exist and equal some real number $f'(a)$.

(a) Let (h_i) be an arbitrary sequence of points in $\mathbb{R}\backslash\{0\}$ such that $h_i \to 0$. Then $(a+h_i)$ is a sequence of points in $\mathbb{R}\backslash\{a\}$ such that $a+h_i \to a$. Ergo by definition of the limit of functions, the limit of the sequence $\left(\frac{f(a+h_i)-f(a)}{a+h_i-a}\right)$ is $f'(a)$. Since (h_i) was arbitrary, we have proved that $\lim_{h_i \to 0} \frac{f(a+h_i)-f(a)}{h_i} = f'(a)$.

(b) Notice that $\frac{f(a+h)-f(a-h)}{2h} = \frac{1}{2}\frac{f(a+h)-f(a)+f(a)-f(a-h)}{h} = \frac{1}{2}\left[\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right]$. We already know that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$. Moreover, if (h_i) is any sequence of nonzero points such that $h_i \to 0$, then $(-h_i)$ is a sequence of nonzero points tending to 0, so since $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$, we must have $\lim_{i\to\infty}\frac{f(a+-h_i)-f(a)}{-h_i} = f'(a)$ as a limit of sequences. But this implies that $\lim_{i\to\infty}\frac{f(a)-f(a-h_i)}{h_i} = f'(a)$. Since (h_i) was arbitrary, we now have a limit of functions $\lim_{h\to 0}\frac{f(a)-f(a-h)}{h} = f'(a)$. Ergo in total we have

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} &= \lim_{h\to 0}\frac{1}{2}\left[\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right] \\
&= \frac{1}{2}[f'(a)+f'(a)] \\
&= f'(a)
\end{aligned}$$

- (28.16) First, suppose $f'(a)$ exists. Then for $x \neq a$, let $\epsilon(x) = f'(a) - \frac{f(x)-f(a)}{x-a}$, and define $\epsilon(a) = 0$. Then $\epsilon(x)$ clearly satisfies the relationship given, and $\lim_{x\to a}\epsilon(x) = \lim x \to a[f'(a) - \frac{f(x)-f(a)}{x-a}] = f'(a) - f'(a) = 0$. Conversely, suppose there is a number $f'(a)$ and a function $\epsilon(x)$ on I with $\lim_{x\to a}\epsilon(x) = 0$ such that the relationship $f(x) - f(a) = (x-a)[f'(a)-\epsilon(x)]$ holds. Then when $x \neq a$, we may divide both sides by $x-a$, so when $x \neq a$, we have $f'(a) - \epsilon(x) = \frac{f(x)-f(a)}{x-a}$. But as $x \to a$, the limit of the left hand side is $f'(a)$, so $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and therefore the derivative of f at a exists.