Homework 10 Solutions

MTH 320

2. - Notice $f_n(0) = 0 = f_n(1)$ for all n. For $x \neq 0, 1$, let $\alpha = \frac{1}{1-x}$. Then

$$\begin{aligned} \lim_{n\to\infty} nx(1-x)^n &= \lim_{n\to\infty} x \cdot \frac{n}{\alpha^n} \\ &= x \lim_{n\to\infty} \frac{n}{\alpha^n} \\ &= x(0) \\ &= 0 \end{aligned}$$

   where the limit follows from the fact that, by Exercise 9.13, if $\alpha > 1$, $\lim_{n\to\infty} \frac{\alpha^n}{n} = \infty$. So f_n converges pointwise to $f \equiv 0$ on $[0, 1]$.

   - Since $|f_n(x)| < \frac{1}{\sqrt{n+1}}$, this sequence converges pointwise to 0 on $[0, 1]$.
   - Since $f_n(\pi) = (-1)^n$ for all n, we conclude this sequence of functions does not converge pointwise on $[0, 2\pi]$.
   - Since $f_n(1) = n^2$ for all n, this sequence does not converge pointwise on $[0, 1]$.

3. Problems from Ross.

   (23.1) (a)Clearly $\sum n^2 x^n$ diverges when $x \neq 0$, so the radius of convergence is $R = 0$ and the interval of convergence is $\{0\}$.

   (c) Let $\sum a_n x^n = \sum (\frac{2^n}{n^2}) x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{2n^2}{(n+1)^2}| \to 2 = \beta$ as $n \to \infty$, so the radius of convergence is $R = \frac{1}{2}$. The endpoints of the interval of convergence are $\pm\frac{1}{2}$; we observe that $f(\frac{1}{2}) = \sum \frac{1}{n^2}$ and $f(-\frac{1}{2}) = \sum \frac{(-1)^n}{n^2}$, both of which converge. Thus the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

   (e) Let $\sum a_n x^n = \sum (\frac{2^n}{n!}) x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{2}{n+1}| \to 0$ as $n \to \infty$. Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $\mathbb{R}$.

   (g) Let $\sum a_n x^n = \sum (\frac{3^n}{n \cdot 4^n}) x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{3n}{4(n+1)}| \to \frac{3}{4} = \beta$ as $n \to \infty$. Therefore the radius of convergence is $R = \frac{4}{3}$. The endpoints of the interval of convergence are $\pm\frac{4}{3}$; we observe that $f(\frac{4}{3}) = \sum \frac{1}{n}$, which diverges, and $f(-\frac{4}{3}) = \sum \frac{(-1)^n}{n}$, which converges. So the interval of convergence is $[-\frac{4}{3}, \frac{4}{3})$.

   (23.4) (a) Notice that $|a_n|^{\frac{1}{n}}$ is equal to $\frac{2}{5}$ if n is odd and $\frac{6}{5}$ if n is odd. We see that $\limsup |a_n|^{\frac{1}{n}}$ is $\frac{6}{5}$ and $\liminf |a_n|^{\frac{1}{n}}$ is $\frac{4}{5}$. Similarly $\frac{a_{n+1}}{a_n}$ is either $\frac{2}{5}\left(\frac{1}{3}\right)^n$ if n is even and $\frac{6}{5}(3)^n$ if n is odd, so $\limsup |\frac{a_{n+1}}{a_n}| = \infty$ and $\liminf |\frac{a_{n+1}}{a_n}| = 0$.

(b) No, by the Ratio Test. In both cases $\limsup |a_n|^{\frac{1}{n}} = \frac{6}{5} > 1$.

(c) The radius of convergence R is $\frac{5}{6}$. Substituting $x = \frac{5}{6}$ gives a power series $\sum b_n$ such that $b_n = 1$ if n is even, which certainly cannot converge. Similar, substituting $x = -\frac{5}{6}$ gives a series $\sum c_n$ such that $c_n = (-1)^n$ if n is even, which also cannot converge. So the interval of convergence is $(-\frac{5}{6}, \frac{5}{6})$.

(23.5) Let $\sum a_n x^n$ be a power series with radius of convergence R.

(a) If the coefficients a_n are all integers and infinitely many are nonzero, then infinitely many $|a_n|^{\frac{1}{n}}$ are greater than or equal to 1. Therefore $\sup\{|a_n|^{\frac{1}{n}} : n > N\}$ is at least one for any N. Ergo the $\beta = \limsup |a_n|^{\frac{1}{n}} \geq 1$, so $R = \frac{1}{\beta} \leq 1$.

(b) Suppose that $\limsup |a_n| > 0$. Choose a constant c such that $0 < c < \limsup |a_n|$. Then $\sup\{a_n : n > N\} > \limsup |a_n| > c$ for all N. Therefore some subsequence (a_{n_k}) of (a_n) has the property that $|a_{n_k}| > c$ for all k. Therefore $|a_{n_k}|^{\frac{1}{n_k}} \geq c^{\frac{1}{n_k}}$. Since c is a constant, $\lim_{k\to\infty} c^{\frac{1}{n_k}} = 1$, so $\limsup |a_{n_k}|^{\frac{1}{n_k}} \geq 1$. This implies that the limit supremum of the entire sequence is greater than or equal to one, i.e. $\beta = \limsup |a_n|^{\frac{1}{n}} \geq 1$. Therefore $R = \frac{1}{\beta} \leq 1$.

(24.2) (a) The pointwise limit of $f_n(x) = \frac{x}{n}$ on $[0, \infty)$ is $f(x) \equiv 0$.

(b) Let $\epsilon > 0$. Choose N such that $\frac{1}{N} < \epsilon$. For any $x \in [0, 1]$ and $n > N$, we have $|f_n(x) - f(x)| = \frac{x}{n} < \frac{1}{n} < \epsilon$. So (f_n) converges uniformly on $[0, 1]$.

(c) Let $\epsilon = 1$. Given any n a natural number, choose $x = n$. Then $|f_n(x) - f(x)| = 1$. So there is no N such that $n > N$ implies $|f_n(x) - f(x)| < 1$. Ergo (f_n) does not converge uniformly on $[0, \infty)$.

(24.3) (a) The pointwise limit of $f_n(x) = \frac{1}{1+x^n}$ is

$$f(x) = \begin{cases} 1 & x \in [0, 1) \\ \frac{1}{2} & x = 1 \\ 0 & x \in (1, \infty) \end{cases}$$

(b) No; each (f_n) is continuous on $[0, 1]$ but the limit f is not continuous on $[0, 1]$, hence (f_n) cannot converge uniformly.

(c) No, similarly.

(24.4) (a) the pointwise limit of $f_n(x) = \frac{x^n}{1+x^n} = 1 - \frac{1}{1+x^n}$ is

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ \frac{1}{2} & x = 1 \\ 1 & x \in (1,\infty) \end{cases}$$

(b)&(c) Again because f is discontinuous at $x = 1$, (f_n) cannot converge uniformly to f either on $[0,1]$ or on $[0,\infty)$.

(24.10)(a) Let $\epsilon > 0$. Choose N_1 such that $n > N_1$ and $x \in S$ implies that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Furthermore choose N_2 such that $n > N_2$ implies $|g_n(x) - g(x)| < \frac{\epsilon}{2}$. Then $n > \max\{N_1, N_2\}$ implies that

$$|f_n(x) + g_n(x) - f(x) + g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) We proceed to 24.11.

(24.11)(a) Indeed, obvious.

(b) The sequence $(f_n g_n) = (\frac{x}{n})$ does not converge uniformly on $\mathbb{R}$ by exercise (24.2)(c).