Homework 10 Solutions

MTH 320

2. • Notice $f_n(0) = 0 = f_n(1)$ for all $n$. For $x \neq 0, 1$, let $\alpha = \frac{1}{1-x}$. Then

$$\lim_{n \to \infty} nx(1-x)^n = \lim_{n \to \infty} x \cdot \frac{n}{\alpha^n} = x \lim_{n \to \infty} \frac{n}{\alpha^n} = x(0) = 0$$

where the limit follows from the fact that, by Exercise 9.13, if $\alpha > 1$, $\lim_{n \to \infty} \alpha^n = \infty$. So $f_n$ converges pointwise to $f \equiv 0$ on $[0, 1]$.

• Since $|f_n(x)| < \frac{1}{\sqrt{n+1}}$, this sequence converges pointwise to 0 on $[0, 1]$.

• Since $f_n(\pi) = (-1)^n$ for all $n$, we conclude this sequence of functions does not converge pointwise on $[0, 2\pi]$.

• Since $f_n(1) = n^2$ for all $n$, this sequence does not converge pointwise on $[0, 1]$.

3. Problems from Ross.

(23.1) (a) Clearly $\sum n^2 x^n$ diverges when $x \neq 0$, so the radius of convergence is $R = 0$ and the interval of convergence is $\{0\}$.

(c) Let $\sum a_n x^n = \sum (\frac{2^n}{3^n}) x^n$. We observe that $\limsup |\frac{a_{n+1}}{a_n}| = |\frac{2^{n+1}}{3^{n+1}}| = 2 = \beta$ as $n \to \infty$, so the radius of convergence is $R = \frac{1}{2}$. The endpoints of the interval of convergence are $\pm \frac{1}{2}$; we observe that $f(\frac{1}{2}) = \sum \frac{1}{n^2}$ and $f(-\frac{1}{2}) = \sum (\frac{-1}{n^2})$, both of which converge. Thus the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

(e) Let $\sum a_n x^n = \sum (\frac{2^n}{3^n}) x^n$. We observe that $\limsup |\frac{a_{n+1}}{a_n}| = \frac{2}{3} \to 0$ as $n \to \infty$. Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $\mathbb{R}$.

(g) Let $\sum a_n x^n = \sum (\frac{3^n}{4^n}) x^n$. We observe that $\limsup |\frac{a_{n+1}}{a_n}| = |\frac{3n}{4(n+1)}| \to \frac{3}{4} = \beta$ as $n \to \infty$. Therefore the radius of convergence is $R = \frac{4}{3}$. The endpoints of the interval of convergence are $\pm \frac{4}{3}$; we observe that $f(\frac{4}{3}) = \sum \frac{1}{n}$, which diverges, and $f(-\frac{4}{3}) = \sum (\frac{-1}{n})$, which converges. So the interval of convergence is $[-\frac{4}{3}, \frac{4}{3}]$.

(23.4) (a) Notice that $|a_n|^\frac{1}{n}$ is equal to $\frac{2}{5}$ if $n$ is odd and $\frac{6}{5}$ if $n$ is odd. We see that $\limsup |a_n|^\frac{1}{n}$ is $\frac{6}{5}$ and $\liminf |a_n|^\frac{1}{n}$ is $\frac{4}{5}$. Similarly $\frac{a_{n+1}}{a_n}$ is either $\frac{2}{5}$ or $\frac{1}{3}$ if $n$ is even and $\frac{6}{5}(3)^n$ if $n$ is odd, so $\limsup |\frac{a_{n+1}}{a_n}| = \infty$ and $\liminf |\frac{a_{n+1}}{a_n}| = 0$. 

(b) No, by the Ratio Test. In both cases \( \limsup |a_n|^\frac{1}{n} = \frac{6}{5} > 1 \).

(c) The radius of convergence \( R \) is \( \frac{5}{6} \). Substituting \( x = \frac{5}{6} \) gives a power series \( \sum b_n \) such that \( b_n = 1 \) if \( n \) is even, which certainly cannot converge. Similar, substituting \( x = -\frac{5}{6} \) gives a series \( \sum c_n \) such that \( c_n = (-1)^n \) if \( n \) is even, which also cannot converge. So the interval of convergence is \( (-\frac{5}{6}, \frac{5}{6}) \).

(23.5) Let \( \sum a_n x^n \) be a power series with radius of convergence \( R \).

(a) If the coefficients \( a_n \) are all integers and infinitely many are nonzero, then infinitely many \( |a_n|^\frac{1}{n} \) are greater than or equal to 1. Therefore \( \sup\{ |a_n|^\frac{1}{n} : n > N \} \) is at least one for any \( N \). Ergo the \( \beta = \limsup |a_n|^\frac{1}{n} \geq 1 \), so \( R = \frac{1}{\beta} \leq 1 \).

(b) Suppose that \( \limsup |a_n| > 0 \). Choose a constant \( c \) such that \( 0 < c < \limsup |a_n| \). Then \( \sup\{ |a_n| : n > N \} > \limsup |a_n| > c \) for all \( N \). Therefore some subsequence \( (a_{n_k}) \) of \( (a_n) \) has the property that \( |a_{n_k}| > c \) for all \( k \). Therefore \( |a_{n_k}|^\frac{1}{n_k} \geq c^\frac{1}{n_k} \). Since \( c \) is a constant, \( \lim_{k \to \infty} c^\frac{1}{n_k} = 1 \), so \( \limsup |a_{n_k}|^\frac{1}{n_k} \geq 1 \). This implies that the limit supremum of the entire sequence is greater than or equal to one, i.e. \( \beta = \limsup |a_n|^\frac{1}{n} \geq 1 \). Therefore \( R = \frac{1}{\beta} \leq 1 \).

(24.2) (a) The pointwise limit of \( f_n(x) = \frac{x}{n} \) on \([0, \infty) \) is \( f(x) \equiv 0 \).

(b) Let \( \epsilon > 0 \). Choose \( N \) such that \( \frac{1}{N} < \epsilon \). For any \( x \in [0, 1] \) and \( n > N \), we have \( |f_n(x) - f(x)| = \frac{x}{n} < \frac{1}{n} < \epsilon \). So \( (f_n) \) converges uniformly on \([0, 1] \).

(c) Let \( \epsilon = 1 \). Given any \( n \) a natural number, choose \( x = n \). Then \( |f_n(x) - f(x)| = 1 \). So there is no \( N \) such that \( n > N \) implies \( |f_n(x) - f(x)| < 1 \). Ergo \( (f_n) \) does not converge uniformly on \([0, \infty) \).

(24.3) (a) The pointwise limit of \( f_n(x) = \frac{1}{1+x^n} \) is

\[
    f(x) = \begin{cases} 
        1 & x \in [0, 1) \\
        \frac{1}{2} & x = 1 \\
        0 & x \in (1, \infty)
    \end{cases}
\]

(b) No; each \((f_n)\) is continuous on \([0, 1]\) but the limit \( f \) is not continuous on \([0, 1]\), hence \((f_n)\) cannot converge uniformly.

(c) No, similarly.
(24.4) (a) the pointwise limit of \( f_n(x) = \frac{x^n}{1+x^n} = 1 - \frac{1}{1+x^n} \) is

\[
f(x) = \begin{cases} 
0 & x \in [0, 1) \\
\frac{1}{2} & x = 1 \\
1 & x \in (1, \infty)
\end{cases}
\]

(b) & (c) Again because \( f \) is discontinuous at \( x = 1 \), \((f_n)\) cannot converge uniformly to \( f \) either on \([0, 1]\) or on \([0, \infty)\).

(24.10) (a) Let \( \epsilon > 0 \). Choose \( N_1 \) such that \( n > N_1 \) and \( x \in S \) implies that \(|f_n(x) - f(x)| < \frac{\epsilon}{2}\). Furthermore choose \( N_2 \) such that \( n > N_2 \) implies \(|g_n(x) - g(x)| < \frac{\epsilon}{2}\). Then \( n > \text{max}\{N_1, N_2\} \) implies that

\[
|f_n(x) + g_n(x) - f(x) + g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

(b) We proceed to 24.11.

(24.11) (a) Indeed, obvious.

(b) The sequence \((f_n g_n) = (\frac{x}{n})\) does not converge uniformly on \( \mathbb{R} \) by exercise (24.2)(c).