Homework 5 Solutions

MTH 310

13,14,15,16,25,26

- (Section 3.2 Problem 13)

  (a) $S \cap T$ is in general a subring. Since both S and T are closed under subtraction and multiplication, the set of elements that are in both is as well.

  (b) $S \cup T$ need not be a subring of R. Consider the subrings $\{0, 3\}$ and $\{0, 2, 4\}$ of $\mathbb{Z}_6$. Their union $\{0, 2, 3, 4\}$ is not a subring; it is not additively closed.

- (Section 3.2 Problem 14) Let R be an integral domain. Let e be a nonzero idempotent, so that $e^2 = e$. Since multiplicative cancellation holds in integral domains, it follows that $e = 1$. So the only idempotents in an integral domain are 0 and 1.

- (Section 3.2 Problem 15) (a) We recall that multiplicative inverses, if they exist, are unique, and observe that $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a(1)a^{-1} = aa^{-1} = 1$. Similarly $(b^{-1}a^{-1})ab = 1$.

  (b) We can get this from any pair of matrices which fail to commute.

- (Section 3.2 Problem 16) This is false. For example, 0 is not a unit.

- (Section 3.2 Problem 25) (a) Consider $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. This is a ring with identity, namely $1_R = 1$. Let S be the subring $\{0, 3\}$. Then $0(3) = 0 = 3(0)$ and $3(3) = 3$, so $3 = 1_S$ is a multiplicative identity.

  (b) Let $S \subset R$ be such that S and R are both integral domains. Suppose that $1_S \neq 1_R$. Let a be a nonzero element of S. Then in R we have $1_R(a) = a = 1_S(a)$. But R is an integral domain, so multiplicative cancellation is valid, implying that $1_R = 1_S$.

- (Section 3.2 Problem 26) Let S be a subring of a ring R. Consider an element $s \in S$. Then $s + 0_s = s$ and in R $s + 0_R = s$. So $s + 0_R = s + 0_S$, implying that $0_R = 0_S$.