Homework 5 Solutions

MTH 310

- (Section 3.2 Problem 2) We assume associativity and distributivity. Closure under addition and multiplication is clear from the tables. We further observe from the tables that addition commutes (in particular, the addition table is symmetric across the diagonal) and that 0 is a zero element. We further note that $e+e = b+b = c+c = 0$, so every element is its own additive inverse. Finally, we observe that e functions as a multiplicative identity. Hence R is a ring. Multiplication commutes (the multiplication table is symmetric across the diagonal). However R is not a field; we perceive that b and c do not have multiplicative inverses.

- (Section 3.2 Problem 6)(a) Let $R = \{3n :\in \mathbb{Z}\}$. We perceive that $3n+3m = 3(n+m) \in R$, so R is closed under addition. Morevoer we see $3n(3m) = 3(3nm)$, so R is closed under multiplication. In addition, $0 = 3(0)$, so $0 \in R$; finally, if $3n \in R$, then $-3n = 3(-n) \in R$, so R is closed under taking additive inverses. We conclude that R is a subring of $\mathbb{Z}$.

  (b) The argument above with 3 replaced by k throughout shows that $\{kn : n \in \mathbb{Z}\}$ is a subring of $\mathbb{Z}$.

- (Section 6 Problem 11) (a) We wish to show the subset S of $M(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

is a subring of $M(\mathbb{R})$. First, the zero element of $M(\mathbb{R})$ is an element of S by letting $a = b = 0$. Second, we perceive that S is closed under taking additive inverses, since

$$-\begin{pmatrix} a & a \\ b & b \end{pmatrix} = \begin{pmatrix} -a & -a \\ -b & -b \end{pmatrix}.$$

Next, S is closed under addition, since

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} + \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} a+c & a+c \\ b+d & b+d \end{pmatrix}$$

Finally, and most interestingly, S is closed under multiplication since

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \times \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{pmatrix}$$

(b,c) We observe that

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

for any a and b but

$$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}2 & 2\\1 & 1\end{pmatrix}=\begin{pmatrix}3 & 3\\0 & 0\end{pmatrix}$$

Hence J is a right identity but not a left identity.

- (Section 3.1 Problem 17) We let $ab = 0$ in $\mathbb{Z}$ and addition be unchanged. First, the axioms for addition plainly continue to hold. Second, closure of the ring under multiplication is obvious, since $0 \in \mathbb{Z}$. As for associativity, for $a, b, c \in \mathbb{Z}$, we have $a(bc) = a(0) = 0 = (0)c = (ab)c$. Finally, for distributivity, we observe that $a(b+c) = 0 = 0 + 0 = ab + ac$ and likewise $(a + b)c = 0 = 0 + 0 = ac + bc$.

- (Section 3.1 Problem 19) This is a straightforward computation; for example we have $E \cdot D = A$.

- (Section 3.1 Problem 24) We take our operations of addition to be $a \oplus b = a + b - 1$ and $a \odot b = ab - (a + b) + 2$. Clearly this is additively and multiplicatively closed. We check the remaining six ring axioms.

  (2) (Associative Addition) For $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned} a \oplus (b \oplus c) &= a + (b \oplus c) - 1 = (a + b + c - 1) - 1 = a + b + c - 2 \\ (a \oplus b) \oplus c &= (a \oplus b) + c - 1 = (a + b - 1) + c - 1 = a + b + c - 2 \end{aligned}$$

  We perceive that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

  (3) (Commutative Addition) We perceive that $a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$.

  (4) (Additive Identity) We perceive that $a \oplus 1 = a + 1 - 1 = a$ and $1 \oplus a = 1 + a - 1 = a$ for all $a \in \mathbb{Z}$. Hence the additive identity is 1.

  (5) (Additive Inverses) We perceive that for $a \in \mathbb{Z}$, we have $a \oplus (2 - a) = a + (2 - a) - 1 = 1$. So the additive inverse of a is $2 - a$.

  (7) (Associative Multiplication) For $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned} a \odot (b \odot c) &= a \odot (bc - (b + c) + 2) \\ &= a(bc - b - c + 2) - a - (bc - b - c + 2) + 2 \\ &= abc - ab - ac - bc + a + b + c. \\ (a \odot b) \odot c &= (a \odot b)c - a \odot b - c + 2 \\ &= (ab - a - b + 2)c - (ab - a - b + 2) + 2 \\ &= abc - ac - bc - ab + a + b + c. \end{aligned}$$

  We perceive that $a \odot (b \odot c) = (a \odot b) \odot c$.

(8) (Distributivity) We check below that multiplication commutes, so it suffices to show that $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

$$\begin{aligned}
a \odot (b \oplus c) &= a \odot (b + c - 1) \\
&= a(b + c - 1) - a - (b + c - 1) + 2 \\
&= ab + ac - 2a - b - c + 3 \\
(a \odot b) \oplus (a \odot c) &= (ab - a - b + 2) \oplus (ac - a - c + 2) \\
&= (ab - a - b + 2) + (ac - a - c + 2) - 1 \\
&= ab + ac - 2a - b - c + 3
\end{aligned}$$

We claim that in addition these operations make $\mathbb{Z}$ into an integral domain. There are three things to check.

(Commutative Multiplication) We observe that $a \odot b = ab - a - b + 2 = ba - b - a + 2 = b \odot a$.

(Multiplicative Identity) We observe that $2 \odot a = 2a - 2 - a + 2 = a = a \odot 2$. So 2 is the multiplicative identity.

(No zero divisors) Suppose that $a \odot b = 1$. Then we have $ab - a - b + 2 = 1$, or $ab - a - b + 1 = 0$. This factors as $(a - 1)(b - 1) = 0$, so we see that either $a = 1$ or $b = 1$. Since 1 is the zero element of $\mathbb{Z}$ with this notion of addition, we see this ring has no product of two nonzero elements equal to zero.

- (Section 3.1 Problem 33) Most of the axioms follow directly from the fact that R and S are rings. Most interestingly, the additive identity of $R \times S$ is $(0_R, 0_S)$, the additive inverse of an element (r, s) in $R \times S$ is $-(r, s) = (-r, -s)$, and if R and S have identity, the identity of $R \times S$ is $(1_R, 1_S)$.

- (Section 3.2 Problem 1) In a general ring we have $(a + b)(a - b) = a^2 - ab + ba - b^2$ and $(a + b)^3 = a^3 + aba + ba^2 + b^2a + a^2b + ab^2 + bab + b^3$. In a commutative ring we have $(a + b)(a - b) = a^2 - b^2$ and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

- (Section 3.2 Problem 3) (a) Four of the idempotents in $M(\mathbb{R})$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}.$$

  This is not a complete list.

  (b) Direct computation shows that the idempotents in $\mathbb{Z}_{12}$ are 1 and 4.

- (Section 3.2 Problem 5) (a) Suppose that 0_1 and 0_2 are both additive identity elements in some ring R. Then for $a \in R$, $a + 0_1 = a = a + 0_2$. But since $a + 0_1 = a + 0_2$, using additive cancellation we perceive that $0_1 = 0_2$. Ergo additive identities are unique.

(b) Let R be a ring with identity. Suppose that 1_a and 1_b are both multiplicative identities in R. Then since 1_a is a multiplicative identity, $1_a \cdot 1_b = 1_b$ and similarly since 1_b is a multiplicative identity, $1_a \cdot 1_b = 1_a$. So $1_a = 1_b$.

(c) Suppose that a is a unit a ring with identity R with inverses u and v. Then $au = 1$. But then $v(au) = v(1)$, implying that $(va)u = v$. Since $va = 1$, we see that $u = v$. So a unit can have only one inverse.