3. Let $x \in A \cup (B \cup C)$. Then either $x \in A$ or $x \in B \cup C$. In the first case, we see that since $x \in A$, it follows that $x$ is also an element of $A \cup B$ and $A \cup C$, so $x \in (A \cup B) \cup (A \cup C)$. In the second case, if $x \in B \cup C$, then either $x \in B$ or $x \in C$. If $x \in B$, $x \in A \cup B$; similarly if $x \in C$, $x \in A \cup C$. So $x$ is an element of at least one of $A \cup B$ and $A \cup C$, implying that $x \in (A \cup B) \cup (A \cup C)$. As $x$ is arbitrary, $A \cup (B \cup C) \subseteq (A \cup B) \cup (A \cup C)$.

Conversely suppose $x \in (A \cup B) \cup (A \cup C)$. Then either $x \in A \cup B$ or $x \in A \cup C$. There are two possibilities, $x \in A$ and $x \notin A$. If $x \in A$, then $x \in A \cup (B \cup C)$. If $x \notin A$, then if $x \in A \cup B$, we must have $x \notin B$, and if $x \in A \cup C$, we must have $x \notin C$. So we perceive that $x$ is an element of at least one of $B$ and $C$, implying that $x \in B \cup C$. Hence $x \in A \cup (B \cup C)$. Since $x$ was arbitrary, $(A \cup B) \cup (A \cup C) \subseteq A \cup (B \cup C)$. We conclude that the two sets are equal.

- Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, implying that either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$; if $x \in C$, then $x \in A \cap C$. In either eventuality $x \in (A \cap B) \cup (A \cap C)$. Since $x$ was arbitrary, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Conversely let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, implying that in particular $x \in A$ and $x \in B \cup C$, so $x \in A \cap (B \cup C)$. Similarly if $x \in A \cap C$ then $x \in A \cap (B \cup C)$. Since $x$ was arbitrary, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. We conclude that the two sets are equal.

4. Let

\[ A = \{2x : x \in \mathbb{Z}\} \quad A' = \{x \in \mathbb{Z} : 4|x^2\} \quad A'' = \{x \in \mathbb{Z} : (-1)^x = 1\} \]

First we show $A = A'$. Now if $y \in A$, $y = 2x$ for some integer $x$, so $y^2 = 4x^2$ is divisible by 4. Hence $y \in A'$. So $A \subseteq A'$. In the other direction, suppose $y \notin A$. Then by the Division Algorithm, $y = 2k + 1$ for some $k \in \mathbb{Z}$. Then $y^2 = 4k^2 + 4k + 1$, which is certainly not divisible by 4. So any element which is not in $A$ is also not in $A'$, implying that $A' \subseteq A$. Hence $A = A'$.

Next we show that $A = A''$. First, if $y \in A$, then $y = 2x$ for some integer $x$, so $(-1)^y = (-1)^{2x} = ((-1)^2)^x = 1^x = 1$, so $y \in A'$. Hence $A \subseteq A'$. 
Conversely if \( y \) is not in \( A \), then as previously \( y = 2k + 1 \) for some integer \( k \), and 
\[
(-1)^y = (-1)^{2k+1} = (-1)^{2k}(-1) = (1)(-1) = -1,
\]
so \( y \) is not in \( A'' \). Hence \( A'' \subset A \). So \( A = A' \).

It follows that \( A = A' = A'' \).

5. (a) \( 241 = 17(13) + 3 \).

(b) \( -241 = 17(-14) + 14 \).

(c) \( 0 = 17(0) + 0 \).

6. We observe that by the Division Algorithm any integer \( a \) can be written as one of \( 3q \), \( 3q + 1 \), or \( 3q + 2 \). We look at each of these cases separately.

\[
(3q)^2 = 9q^2 = 3(3q^2) = 3k
\]

\[
(3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3k + 1
\]

\[
(3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1 = 3k + 1
\]

So we see that any integer square \( a^2 \) may be written in the form \( 3k \) or \( 3k + 1 \).

7. First, suppose \( a \) and \( c \) leave the same remainder when divided by \( n \). Then we have \( a = nq_1 + r \) and \( c = nq_2 + r \). In particular \( a - c = nq_1 + r - (nq_2 + r) = n(q_1 - q_2) \). So \( n|(a - c) \).

Conversely, suppose that \( n|(a - c) \). Let \( a - c = nk \). Use the Division Algorithm to write \( a = nq_1 + r_1 \) and \( c = nq_2 + r_2 \) for \( 0 \leq r_1, r_2 < n \). Then we see that

\[
nk = (nq_1 + r_1) - (nq_2 - r_2)
\]

\[
nk = n(q_1 - q_2) + (r_1 - r_2)
\]

\[
n(k + q_2 - q_1) = r_1 - r_2
\]

But \( 0 \leq r_1 < n \) and \( -n < -r_2 \leq 0 \), so in fact \( -n < r_1 - r_2 < n \). But the only integer divisible by \( n \) in that range is \( 0 \). So \( r_1 = r_2 \) as desired.

8. We prove an extended version of the Division Algorithm. If \( b > 0 \), this is just the actual Division Algorithm. So let \( b < 0 \). Given \( a \), use the ordinary Division Algorithm to divide \( -a \) by \(|b|\), obtaining \(-a = |b|q + r \) for some \( 0 \leq r < 0 \). Multiplying by \(-1 \) we see that \( a = bq - r \). If \( r = 0 \) we are done. Otherwise we observe that \( a = bq - r = b(q + 1) + (|b| - r) \). We observe that since \( b < -r \leq 0 \), we have \(|b| + b = 0 < |b| - r < |b| \). So setting \( q' = q + 1 \), we have written \( a = bq' + r \) for \( 0 \leq r < |b| \) as desired. Uniqueness follows by the proof of uniqueness in the Division Algorithm (which did not importantly use the sign of \( b \)).