

① ① $\vec{F}(x,y) = (x^2+y^2, -2xy)$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2x)\vec{k} \quad \left(\begin{array}{l} \text{or scalar curl is} \\ -2y - 2x \end{array} \right)$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(-2xy) = 2x - 2x = 0$$

① ② Suppose that \vec{F} were ∇f for some $f(x,y)$. Then we would have $\text{curl } \vec{F} = \text{curl}(\nabla f) = 0$. As we do not, we see that \vec{F} is not a gradient vector field.

(2)

$$(2) \text{ a) } \int_0^3 \int_{y^2}^9 y \cos(x^2) dx dy = \int_0^9 \int_0^{\sqrt{x}} y \cos(x^2) dy dx$$

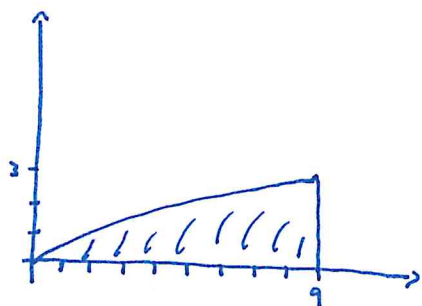
$$= \int_0^9 \left[\frac{1}{2} y^2 \cos(x^2) \right]_0^{\sqrt{x}} dx$$

$$= \int_0^9 \frac{1}{2} x \cos(x^2) dx$$

$$= \frac{1}{4} \sin(x^2) \Big|_0^9$$

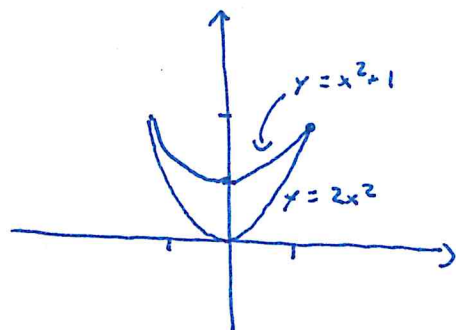
$$= \frac{1}{4} (\sin(81) - 0)$$

$$= \frac{1}{4} \sin(81)$$



$$(6) \int_{-1}^1 \int_{2x^2}^{x^2+1} \int_0^{y+4} y dz dy dx = \int_{-1}^1 \int_{2x^2}^{x^2+1} (y^2 + 4y) dy dx$$

Region projected onto xy-plane



$$= \int_{-1}^1 \left[\frac{1}{3} y^3 + 2y^2 \right]_{2x^2}^{x^2+1} dx$$

$$= \int_{-1}^1 \left(\left[\frac{1}{3} (x^2+1)^3 + 2(x^2+1)^2 \right] - \left[\frac{1}{3} (2x^2)^3 + 2(2x^2)^2 \right] \right) dx$$

$$= \int_{-1}^1 \left(\frac{1}{3} x^6 + x^4 + x^2 + \frac{1}{3} + 2x^4 + 4x^2 + 2 - \frac{8}{3} x^6 - 8x^4 \right) dx$$

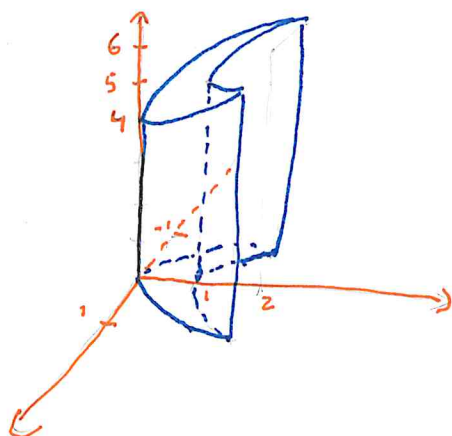
$$= \int_{-1}^1 \left(-\frac{7}{3} x^6 + 5x^4 + 5x^2 + \frac{7}{3} \right) dx$$

$$= \left[-\frac{7}{3} x^7 + x^5 + \frac{5}{3} x^3 + \frac{7}{3} x \right]_{-1}^1$$

$$= -\frac{7}{3} - 2 + \frac{10}{3} + \frac{14}{3}$$

$$= \frac{16}{3}$$

Region



③a $\vec{c}(t) = (\sin t, \cos t, e^t)$

$$\vec{c}'(t) = (\cos t, -\sin t, e^t)$$

$$\|\vec{c}'(t)\| = \sqrt{1 + e^{2t}}$$

$$\text{Arclength} = \int_0^1 \sqrt{1 + e^{2t}} dt$$

[Exercise if you are bored: evaluate this integral using the substitution $u = \sqrt{1 + e^{2t}}$.]

⑥ IF $(x, y, z) = (\sin t, \cos t, e^t)$, we see $\vec{c}'(t) = (\cos t, -\sin t, e^t) = (y, -x, z)$.

So $\vec{c}(t)$ is a flowline of $\vec{F}(x, y, z) = (y, -x, z)$, since $\vec{F}(\vec{c}(t)) = \vec{c}'(t) \quad \forall t$.

④ a) $F(x, y, z) = yz + xy$

$$\frac{\partial F}{\partial x} = y \quad \frac{\partial F}{\partial y} = x + z \quad \frac{\partial F}{\partial z} = y$$

Critical points occur where $\begin{cases} x+z=0 \\ y=0 \end{cases}$. So critical points

are $\{(x, 0, -x) : x \in \mathbb{R}\}$. Note that $F(x, 0, -x) = 0$ for all such critical pts. [In fact by observation these are saddle points.]

⑥ $\nabla F = (y, x+z, y)$

$$g_1(x, y, z) = xy \quad g_2(x, y, z) = y^2 + z^2$$

$$\nabla g_1 = (y, x, 0) \quad \nabla g_2 = (0, 2y, 2z)$$

$$\begin{cases} y = \lambda_1 y + 0 \\ x+z = \lambda_1 x + 2\lambda_2 y \\ y = 2\lambda_2 z \\ xy = 1 \\ y^2 + z^2 = 1 \end{cases}$$

Note $xy=1 \Rightarrow x, y \neq 0$.

So $y = \lambda_1 y \Rightarrow \lambda_1 = 1$.

Now $x+z = x+2\lambda_2 y \Rightarrow z = 2\lambda_2 y$.

So $y = 2\lambda_2 z = 4\lambda_2^2 y \Rightarrow 1 = 4\lambda_2^2$

$\Rightarrow \lambda_2 = \pm \frac{1}{2}$

If $\lambda_2 = \frac{1}{2}$ then $y = 2\lambda_2 z \Rightarrow y = z$,

so since $y^2 + z^2 = 1$, either $y = z = \frac{1}{\sqrt{2}}$ or $y = z = -\frac{1}{\sqrt{2}}$. Likewise if $\lambda_2 = -\frac{1}{2}$ we have $y = -z = \frac{1}{\sqrt{2}}$ or $y = -z = -\frac{1}{\sqrt{2}}$.

So the points at which a max or min could occur

are $a = (\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $c = (\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$b = (-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ $d = (-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

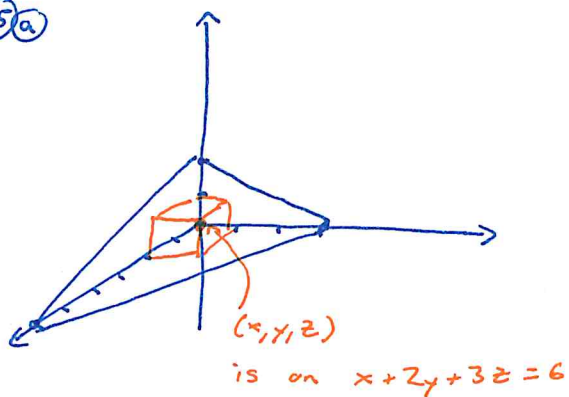
$f(a) = \frac{1}{2} + 1 = \frac{3}{2}$ $f(c) = -\frac{1}{2} + 1 = \frac{1}{2}$

Absolute max = $\frac{3}{2}$

$f(b) = \frac{1}{2} + 1 = \frac{3}{2}$ $f(d) = -\frac{1}{2} + 1 = \frac{1}{2}$

Absolute min = $\frac{1}{2}$

(5a)



$$\text{Volume} = xyz = (6-2y-3z)yz$$

$$V(y,z) = 6yz - 12y^2z - 3z^2y$$

$$\frac{\partial V}{\partial y} = 6z - 24yz - 3z^2$$

$$\frac{\partial V}{\partial z} = 6y - 12y^2 - 6yz$$

$$0 = 6z - 24yz - 3z^2$$

$$= z(6 - 24y - 3z)$$

$$= 3z(2 - 8y - z)$$

$$z=0 \quad z=2-8y$$

$$0 = 6y - 12y^2 - 6yz$$

$$= 6y(1 - 2y - z)$$

$$y=0 \quad z=1-2y$$

$$2-8y = 1-2y \quad z = 1 - \frac{2}{6} = \frac{2}{3}$$

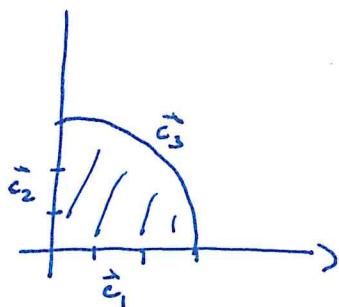
$$1 = 6y$$

$$\frac{1}{6} = y$$

$$V\left(\frac{1}{6}, \frac{2}{3}\right) = \left(6 - \frac{1}{3} - 2\right) \frac{1}{6} \left(\frac{2}{3}\right) = \frac{11}{3} \left(\frac{1}{6}\right) \left(\frac{2}{3}\right) = \frac{11}{9}$$

Note that this has to be the maximum; the region is closed and bounded, and at any ^{other} critical pt or along any part of the bdy we get 0.

3(b) $F(x, y) = xy^2$



"Interior" Critical Pts

$$\frac{\partial F}{\partial x} = y^2 \quad \frac{\partial F}{\partial y} = 2xy \quad x=y=0 \quad (0,0)$$

Boundary

$$\vec{c}_1(t) = (t, 0) \quad F(\vec{c}_1(t)) \equiv 0$$

$$0 \leq t \leq 3$$

$$\vec{c}_2(t) = (0, t) \quad F(\vec{c}_2(t)) \equiv 0$$

$$0 \leq t \leq 3$$

$$\vec{c}_3(t) = (\cos t, \sin t)$$

$$0 \leq t \leq \frac{\pi}{2}$$

$$F(\vec{c}_3(t)) = \cos t \sin^2 t = \cos t - \cos^3 t$$

" $g(t)$

$$g'(t) = -\sin t + 2\sin t \cos^2 t$$

$$= \sin t (2\cos^2 t - 1)$$

$$\sin t = 0$$

$$t = 0$$

$$\text{also } (1, 0)$$

$$2\cos^2 t - 1 = 0$$

$$\cos^2 t = \frac{1}{2}$$

$$t = \frac{\pi}{4}$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Absolute Max $F\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}$

Absolute Min 0

⑥ a) The normal vector to the tangent plane of the ellipsoid at any point is ∇F where $F(x, y, z) = x^2 + 2y^2 + 3z^2$ at that point. Two planes are parallel if their normal vectors are scalar multiples.

So this is the question: at what points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ is ∇F equal to $\lambda(3, -1, 3)$ for some $\lambda \neq 0$?

$$\nabla F(x, y, z) = (2x, 4y, 6z)$$

$$\begin{cases} 2x = 3\lambda \\ 4y = -\lambda \\ 6z = 3\lambda \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases}$$

$$x^2 + 2y^2 + 3z^2 = 1$$

$$\left(\frac{3\lambda}{2}\right)^2 + 2\left(-\frac{\lambda}{4}\right)^2 + 3\left(\frac{\lambda}{2}\right)^2 = 1$$

$$\frac{9\lambda^2}{4} + \frac{\lambda^2}{8} + \frac{3\lambda^2}{4} = 1$$

$$25\lambda^2 = 8$$

$$\lambda = \pm \frac{2\sqrt{2}}{5}$$

Therefore $\Rightarrow (x, y, z) = \pm \left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5} \right)$

6 (b) $F(x, y) = x^2 + y^2 - 2x - 4y$

$$\nabla F(x, y) = (2x - 2, 2y - 4)$$

$$\nabla F(-1, 3) = (-4, 2)$$

Gradient vector points in the direction of
Fastest increase.