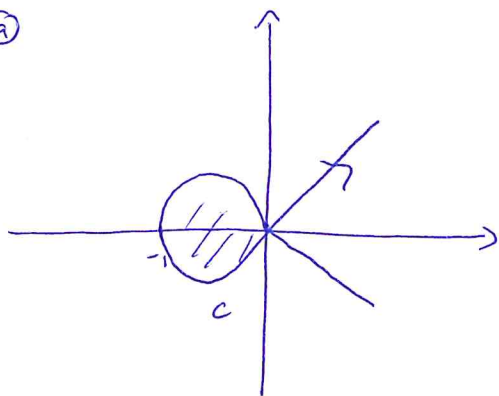


① a



$$\vec{c}(t) = (t^2 - 1, t^3 - t) \quad t \in [-1, 1]$$

By Green's Thm,

$$\text{Area} = \int_c x dy$$

$$= \int_{-1}^1 (t^2 - 1)(3t^2 - 1) dt$$

$$= \int_{-1}^1 (3t^4 - 3t^2 - t^2 + 1) dt$$

$$= 2 \int_0^1 (3t^4 - 4t^2 + 1) dt$$

$$= 2 \left( \frac{3}{5} - \frac{4}{3} + 1 \right)$$

$$= \frac{8}{15}$$

⑥  $\vec{F}(x, y, z) = (2xyz + \sin x, x^2z, x^2y) = \nabla F$  for  $F(x, y, z) = x^2yz - \cos x$

So  $\int_c \vec{F} \cdot d\vec{s} = F(\vec{c}(1)) - F(\vec{c}(0))$

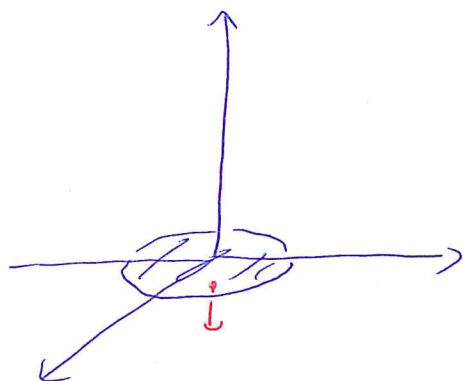
$$= F(e, 6, -1) - F(1, 1, -7)$$

$$= -6e - \cos(e) - (-7 - \cos(1))$$

$$= 7 - 6e + \cos(1) - \cos(e)$$

(2)

② a  $\vec{F}(x, y, z) = (z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2z)$



$\vec{n} = (0, 0, -1)$  everywhere

$\vec{F}(x, y, 0) = (0, \frac{1}{3}y^3, y^2)$

$\vec{F} \cdot \vec{n} = -y^2$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \int_0^{2\pi} \int_0^1 (-r^2 \sin^2 \theta) r \, dr \, d\theta$$

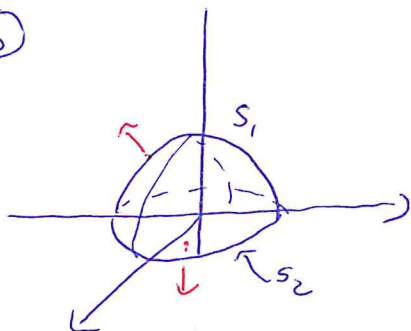
$$= \int_0^{2\pi} -\frac{1}{4} \sin^2 \theta \, d\theta$$

$$= -\frac{1}{4} \int_0^{2\pi} \left[ \frac{1}{2} - \frac{1}{2} \cos 2\theta \right] d\theta$$

$$= -\frac{1}{4} (\pi)$$

$$= -\frac{\pi}{4}$$

⑥



Notice that  $S_1 \cup S_2$  is the boundary of half a solid ball  $W$ .

$$\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F} \, dV$$

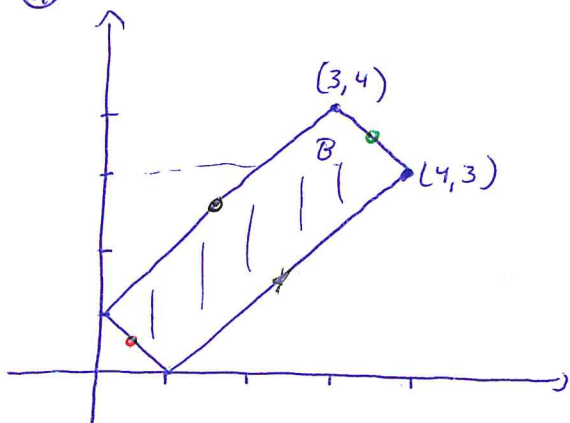
$$\iint_{S_1} \vec{F} \cdot d\vec{S} - \frac{\pi}{4} = \iiint_W (z^2 + y^2 + x^2) \, dV$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \frac{\pi}{4} + \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$$

$$= \frac{\pi}{4} + 2\pi (1) \left(\frac{1}{5}\right)$$

$$= \frac{13\pi}{20}$$

③ a



$$\text{Let } u = x + y$$

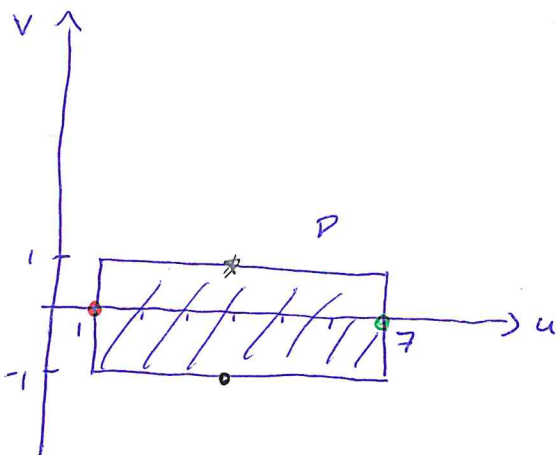
$$v = x - y$$

$$\text{Then } 2x = u + v \Rightarrow x = \frac{u+v}{2}$$

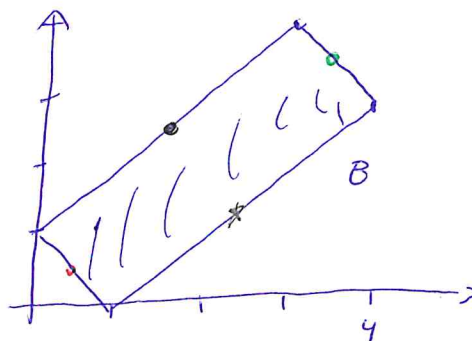
$$2y = u - v \Rightarrow y = \frac{u-v}{2}$$

$$\text{Let } T(u, v) = \left( \frac{u+v}{2}, \frac{u-v}{2} \right).$$

Then we have



$T$   
 $\curvearrowright$   $\mathbb{R}^2$



⑥ The Jacobian of  $T$  is  $|\det(DT(u, v))| = \left| \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}.$

$$\iint_B (x+y) dA = \int_{-1}^1 \int_{\frac{1}{2}}^{\frac{7}{2}} u du dv = \int_{-1}^1 7 dv = \frac{1}{2} (49 - 1) = 24$$

$$\textcircled{4} f(x, y) = (x^2 + 2y, xy^3 - 1) \quad g(x, y) = (\ln x, e^y, x + y)$$

$$\vec{D}f(x, y) = \begin{pmatrix} 2x & 2 \\ y^3 & 3xy^2 \end{pmatrix} \quad \vec{D}g(x, y) = \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & e^y \\ 1 & 1 \end{pmatrix}$$

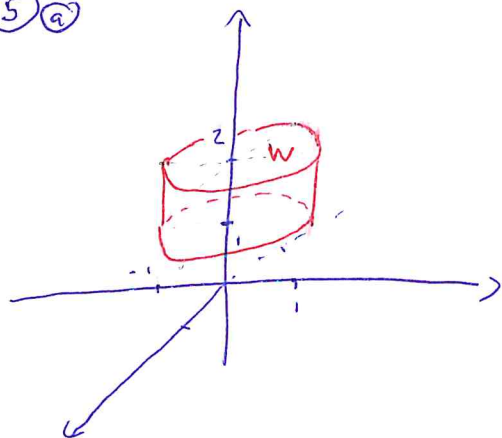
$$\textcircled{b} \quad \vec{D}(g \circ f)(1, 0) = \vec{D}g(f(1, 0)) \cdot \vec{D}f(1, 0)$$

$$= \vec{D}g(1, -1) \cdot \vec{D}f(1, 0)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 2 \end{pmatrix}$$

5a



$$\delta(x, y, z) = (x^2 + y^2)z^2$$

$$\text{Mass} = \iiint_W \delta(x, y, z) dV$$

$$= \int_1^2 \int_0^{2\pi} \int_0^1 r^2 z^2 (r dr d\theta dz)$$

$$= 2\pi \cdot \frac{1}{4} \int_1^2 z^2 dz$$

$$= \frac{\pi}{2} \cdot \frac{1}{3} [8 - 1]$$

$$= \frac{7\pi}{6}$$

⑥ By symmetry, the  $x$  and  $y$  coordinates of the center of mass are both zero. It remains to determine the  $z$  coordinate.

$$\bar{z} = \frac{1}{m} \cdot \iiint_W z(\delta(x, y, z)) dV$$

$$= \frac{6}{7\pi} \int_1^2 \int_0^{2\pi} \int_0^1 r^3 z^3 dr d\theta dz$$

$$= \frac{6}{7\pi} (2\pi) \left(\frac{1}{4}\right) \cdot \int_1^2 z^3 dz$$

$$= \frac{3}{7} \left[ \frac{1}{4} (16 - 1) \right]$$

$$= \frac{3(15)}{28}$$

$= \frac{45}{28}$  } Sanity check: We would expect a number between  $1\frac{1}{2}$  and  $2$ , and that's what we have.

$$\left(0, 0, \frac{45}{28}\right)$$

⑥

$$⑥ \quad F(x, y, z) = xe^z + y \cos x$$

$$\nabla F(x, y, z) = (e^z - y \sin x, \cos x, xe^z)$$

⑦ In the direction of  $\nabla F(0, 1, 5) = (e^5 - 0, 1, 0) = (e^5, 1, 0)$ .

⑧ The normal vector to the plane is  $\nabla F(0, 2, 7) = (e^7, 1, 0)$ .

So the tangent plane is

$$e^7(x - 0) + 1(y - 2) + 0(z - 7) = 0$$

$$\boxed{e^7 x + y = 2}$$

⑦ ②  $\lim_{(x,y) \rightarrow (0,0)} \frac{(x-y)^2}{x^2+y^2}$

If  $x=y$ , this is  $\lim_{x \rightarrow 0} \frac{0}{2x^2} = 0.$

If  $x=-y$ , this is  $\lim_{x \rightarrow 0} \frac{4x^2}{2x^2} = 2.$

The limit does not exist.

⑥ By continuity,  $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy}}{x+1} = \frac{e^0}{1} = 1.$

8)  $F(x, y) = y + x \sin y.$

a)  $\frac{\partial F}{\partial x} = \sin y \quad \frac{\partial F}{\partial y} = 1 + x \cos y$

$0 = \sin y$

$\Rightarrow y = n\pi$  for  
some integer  $n$

$0 = 1 + x \cos y$

$-\frac{1}{x} = \cos y$

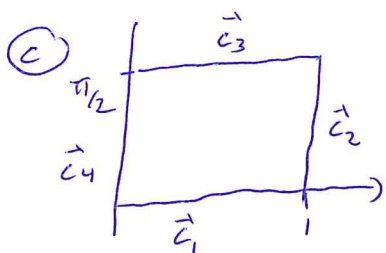
If  $y = n\pi$  for  $n$  even,  $-\frac{1}{x} = 1 \Rightarrow x = -1$

If  $y = n\pi$  for  $n$  odd,  $-\frac{1}{x} = -1 \Rightarrow x = 1$

Critical Points  $(1, n\pi)$   $n$  odd  
 $(-1, n\pi)$   $n$  even

b)  $\begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{vmatrix} = -\cos^2 y < 0$

All of the critical points are neither local maxima nor local minima.



By b),  $\nexists$  interior local max or min. So the max must occur on the boundary.





(9)

$$\bullet \vec{c}_1(t) = (t, 0), \quad t \in [0, 1]$$

$$g(t) = F(t, 0) = 0$$

$$\bullet \vec{c}_2(t) = (1, t), \quad t \in [0, \pi/2]$$

$$g(t) = t + \sin t$$

$$g'(t) = 1 + \cos t$$

$$0 = 1 + \cos t$$

$$-1 = \cos t$$

$$t = \pi \quad \} \text{ Not in region}$$

$$\bullet \vec{c}_3(t) = (t, \frac{\pi}{2}), \quad t \in [0, 1]$$

$$g(t) = \frac{\pi}{2} + t$$

$$g'(t) = 1 \neq 0$$

$$\bullet \vec{c}_4(t) = (0, t), \quad t \in [0, \frac{\pi}{2}]$$

$$g(t) = t$$

$$g'(t) = 1 \neq 0$$

So we check only the four corners

$$F(0, 0) = 0 \quad F(1, \frac{\pi}{2}) = \frac{\pi}{2} + 1 \quad \} \text{ Absolute max.}$$

$$F(0, \frac{\pi}{2}) = \frac{\pi}{2}$$

$$F(1, 0) = 0$$