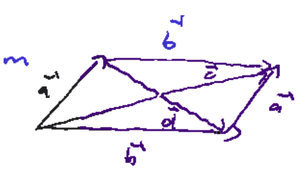
MTH 254H: Quiz 1

1. (5 pts) Show that in any parallelogram the sum of the squares of the lengths of the four sides equals the sum of the squares of the lengths of the two diagonals.

2. (5 pts) The *angle at which two lines intersect* is the angle between their direction vectors; in other words, if $\ell_1(t) = \mathbf{a_1} + t\mathbf{v_1}$ and $\ell_2(t) = \mathbf{a_2} + t\mathbf{v_2}$ and the two lines intersect in a single point, then the angle at which the lines intersect is the angle between $\mathbf{v_1}$ and $\mathbf{v_2}$. The lines $\ell_1(t) = (t+1, 2t-7, 2)$ and $\ell_2(t) = (2t, t+3, t-2)$ intersect at a single point. Find this point, and determine the angle at which these two lines intersect.

① Consider a parallelogram



w/ diagonals

$\vec{c} = \vec{a} + \vec{b}$ and $\vec{d} = \vec{b} - \vec{a}$. We see that

$$\|\vec{c}\|^2 + \|\vec{d}\|^2 = \vec{c} \cdot \vec{c} + \vec{d} \cdot \vec{d}$$

$$= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a})$$

$$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{b} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a}$$

$$= 2\vec{a} \cdot \vec{a} + 2\vec{b} \cdot \vec{b}$$

$$= 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$$

So the sum of the squares of the lengths of the diagonals is the sum of the squares of the lengths of all four sides.

② $\ell_1(t)$ and $\ell_2(t)$ exist when for some values of t and s

$$\begin{cases} t+1 = 2s & ① \\ 2t-7 = s+3 & ② \\ 2 = s-2 & ③ \end{cases}$$

③ $2 = s-2$, $4 = s$

① $t+1 = 2s$, $t+1 = 8$, $t = 7$

② $2(7) - 7 = 4+3$ ✓

The lines intersect at $t=7$ and $s=4$, which is the point $(8, 7, 2)$.

The angle between the lines is the angle between their direction vectors $\vec{v}_1 = (1,2,0)$ and $\vec{v}_2 = (2,1,1)$ is $\theta = \cos^{-1}\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right) = \cos^{-1}\left(\sqrt{\frac{8}{15}}\right)$.