

$$\textcircled{1} h(x, y) = xy^2 - y + \cos(\pi x)$$

$$\textcircled{a} \nabla h(x, y) = (y^2 + \pi(-\sin(\pi x)), 2xy - 1) = (y^2 - \pi \sin(\pi x), 2xy - 1)$$

$$\nabla h(1, 1) = (1 + \pi(-\sin(\pi)), 2 - 1) = (1, 1)$$

If you walk due north at speed one,  $h$  increases at

$$\nabla_{(0,1)} h \text{ has } \nabla h(1, 1) \cdot (0, 1) = (1, 1) \cdot (0, 1) = 1 \text{ m/s.}$$

$\textcircled{b}$  You should walk in the direction of  $\nabla h(1, 1) = (1, 1)$  to ascend the mountain as fast as possible.

$\textcircled{c}$  Suppose  $0 = \nabla_{\vec{u}} h(1, 1) = \nabla h(1, 1) \cdot \vec{u} = (1, 1) \cdot (u_1, u_2) = u_1 + u_2$ .

This implies  $u_1 = -u_2$ . So the two directions in which you can walk to stay at the same height are  $\vec{u} = \pm \left( \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$ .

$[\pm(1, -1) \text{ was also accepted.}]$



$$\textcircled{2} \quad \vec{F}(x, y, z) = (2xy + z \cos y, x^2 - xz \sin y, x \cos y)$$

$$\textcircled{a} \quad \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z \cos y & x^2 - xz \sin y & x \cos y \end{vmatrix}$$

$$= (-x \sin y - (-x \sin y)) \vec{i} + (\cos y - \cos y) \vec{j} + ((2x - z \sin y) - (2x - z \sin y)) \vec{k} \\ = \vec{0}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x} (2xy + z \cos y) + \frac{\partial}{\partial y} (x^2 - xz \sin y) + \frac{\partial}{\partial z} (x \cos y) \\ = 2y + (-xz \cos y) + 0 \\ = 2y - xz \cos y$$

$\textcircled{b} \textcircled{i}$  Yes; observe that we could take  $\vec{F} = \nabla F$  where  $F(x, y, z) = x^2y + xz \cos y$ . [Note that "because  $\text{curl } \vec{F} = \vec{0}$ " is not a correct answer.]

$\textcircled{ii}$  No; if we had  $\vec{F} = \text{curl } \vec{G}$  then we would have  $\text{div}(\vec{F}) = \text{div}(\text{curl } \vec{G}) = 0$ , and we don't.



$$(3a) \quad F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$g(x, y, z) = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

$$\nabla F = \left( -\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2} \right)$$

$$\nabla g = \left( -\frac{2}{x^3}, -\frac{2}{y^3}, -\frac{2}{z^3} \right)$$

$$\begin{cases} -\frac{1}{x^2} = \lambda \left( -\frac{2}{x^3} \right) \\ -\frac{1}{y^2} = \lambda \left( -\frac{2}{y^3} \right) \\ -\frac{1}{z^2} = \lambda \left( -\frac{2}{z^3} \right) \\ \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 1 \end{cases}$$

Note  $x, y, z \neq 0$  everywhere on  $g(x, y, z) = 1$ .

$$\text{So } x^3 = 2\lambda x^2$$

$$x = 2\lambda$$

Likewise  $y = z = 2\lambda$ .

$$\text{So } \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$3 = 4\lambda^2$$

$$\frac{3}{4} = \lambda^2$$

$$\pm \frac{\sqrt{3}}{2} = \lambda$$

$$\Rightarrow (x, y, z) = \pm(\sqrt{3}, \sqrt{3}, \sqrt{3})$$

$$F(\sqrt{3}, \sqrt{3}, \sqrt{3}) = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \text{Global Max}$$

$$F(-\sqrt{3}, -\sqrt{3}, -\sqrt{3}) = -\sqrt{3} \quad \text{Global Min}$$



$$\textcircled{3} \textcircled{b} \quad xy^2z^3 = 2$$

Want to minimize  $F(x, y, z) = x^2 + y^2 + z^2$  (the square of the distance to the origin). But  $y^2 = \frac{2}{xz^3}$ .

$$F(x, z) = x^2 + z^2 + \frac{2}{xz^3}$$

$$\frac{\partial F}{\partial x} = 2x + \frac{-2}{x^2 z^3}$$

$$\frac{\partial F}{\partial z} = 2z + \frac{-6}{xz^4}$$

$$0 = 2x + \frac{-2}{x^2 z^3}$$

$$0 = 2z + \frac{-6}{xz^4}$$

$$\frac{2}{x^2 z^3} = 2x$$

$$\frac{3}{xz^4} = z$$

$$\frac{1}{z^3} = x^3$$

$$3 = z^5(x)$$

$$\frac{1}{z} = x$$

$$3 = z^4$$

$$\pm 4\sqrt{3} = z$$

So  $(x, y, z)$  is one of  $\left(\left(\frac{1}{3}\right)^{1/4}, \left(\frac{4}{3}\right)^{1/4}, 3^{1/4}\right)$   $\left(\left(\frac{1}{3}\right)^{1/4}, -\left(\frac{4}{3}\right)^{1/4}, 3^{1/4}\right)$

$\left(-\left(\frac{1}{3}\right)^{1/4}, \left(\frac{4}{3}\right)^{1/4}, -3^{1/4}\right)$   $\left(-\left(\frac{1}{3}\right)^{1/4}, -\left(\frac{4}{3}\right)^{1/4}, -3^{1/4}\right)$

At each of these points  $(x, y, z)$ ,  $F(x, y, z) = 2\sqrt{3}$ , and the distance to the origin is  $(12)^{1/4}$ .



④ a)  $F(x, y) = 4xy^2 - x^2y^2 - xy^3$

$$\frac{\partial F}{\partial x} = 4y^2 - 2xy^2 - y^3$$

$$\frac{\partial F}{\partial y} = 8xy - 2x^2y - 3xy^2$$

$$0 = 4y^2 - 2xy^2 - y^3$$

$$0 = xy(8 - 2x - 3y)$$

$$= y^2(4 - 2x - y)$$

$$x=0 \quad y=0 \quad 8=2x+3y$$

$$y=0 \quad y=4-2x$$

Options

i)  $y=0$   $(x, 0)$  is always a critical point.

ii)  $x=0, y=4-2x$   $(0, 4)$

iii)  $y=4-2x, 8=2x+3y$   $(1, 2)$   
 $2=4-2x \quad 8=(4-y)+3y$   
 $2x=2 \quad 4=2y$   
 $x=1 \quad 2=y$

Classify

$$D = \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -2y^2 & 8y - 4xy - 3y^2 \\ 8y - 4xy - 3y^2 & 8x - 2x^2 - 6xy \end{vmatrix}$$

At  $(0, 4)$ :  $\frac{\partial^2 F}{\partial x^2} = -2(16) = -32 < 0$

$$D = \begin{vmatrix} -32 & -16 \\ -16 & 0 \end{vmatrix} = -(16)^2 < 0 \quad \text{No local extremum.}$$



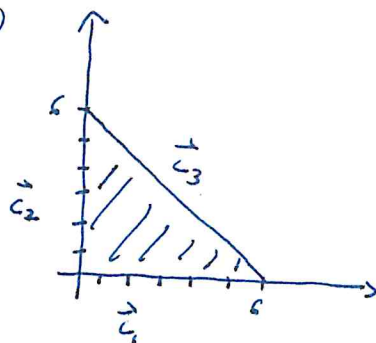
4a) ctd

$$\text{At } (1, 2): \frac{\partial^2 F}{\partial x^2} = -2(4) = -8 < 0$$

$$D = \begin{vmatrix} -8 & -4 \\ -4 & -6 \end{vmatrix} = 48 - 16 = 32$$

} Local maximum

6



Interior critical points: (1, 2), (0, 4)

Boundary

• on  $\vec{c}_1$  and  $\vec{c}_2$ ,  $F \equiv 0$ .

$$\vec{c}_3(t) = (6-t, t) \quad 0 \leq t \leq 6$$

$$\begin{aligned} g(t) &= F(\vec{c}_3(t)) = 4(6-t)t^2 - (6-t)^2 t^2 - (6-t)t^3 \\ &= 24t^2 - 4t^3 - 36t^2 + 12t^3 - t^4 \\ &\quad - 6t^3 + t^4 \\ &= 2t^3 - 12t^2 \end{aligned}$$

$$0 = g'(t) = 6t^2 - 24t$$

$$0 = 6t(t-4)$$

$$t = 4$$

$$(2, 4)$$

$$F(1, 2) = 4(1)(2)^2 - (1)^2(2)^2 - (1)(2)^3 = 16 - 4 - 8 = \boxed{4} \leftarrow \text{Global max}$$

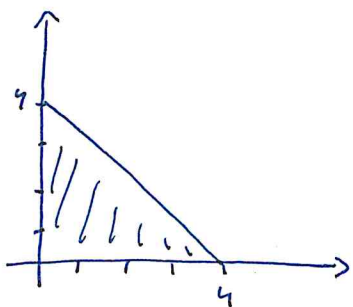
$$F(0, 4) = 0$$

$$F(2, 4) = 4(2)(16) - 4(16) - 2(64) = \boxed{-64} \leftarrow \text{Global min}$$

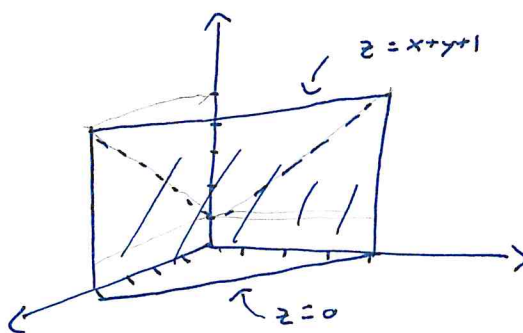


5 a

Projection to  $xy$ -plane



Region in 3-space



$$V = \int_{x=0}^{x=4} \int_{y=0}^{y=4-x} \int_{z=0}^{z=x+y+1} dz dy dx$$

$$= \int_0^4 \int_0^{4-x} (x+y+1) dy dx$$

$$= \int_0^4 \left[ (x+1)(4-x) + \frac{1}{2}(4-x)^2 \right] dx$$

$$= \int_0^4 \left[ 4x - x + 4 - x^2 + 8 - 4x + \frac{1}{2}x^2 \right] dx$$

$$= \int_0^4 \left[ -\frac{1}{2}x^2 - x + 12 \right] dx$$

$$= \left[ -\frac{1}{6}x^3 - \frac{1}{2}x^2 + 12x \right]_0^4$$

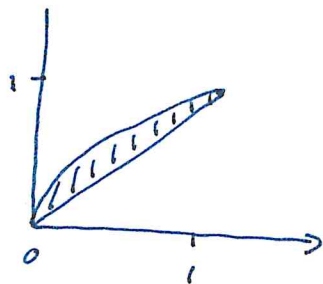
$$= -\frac{32}{3} - 8 + 48$$

$$= -\frac{32}{3} + 40$$

$$= \frac{88}{3}$$



$$\textcircled{5} \textcircled{6} \quad \int_0^1 \int_x^{\sqrt{x}} e^{\frac{x}{y}} dy dx = \int_0^1 \int_{y^2}^y e^{\frac{x}{y}} dx dy$$



$$= \int_0^1 \left[ y e^{\frac{x}{y}} \right]_{y^2}^y dy$$

$$= \int_0^1 y [e - e^y] dy$$

$$= \int_0^1 (ey - ye^y) dy$$

$$= \left[ \frac{1}{2} ey^2 - (ye^y - e^y) \right]_0^1$$

$$= \left[ \frac{1}{2} e - e + e \right] - [0 + 1]$$

$$= \frac{e}{2} - 1$$

y	e <sup>y</sup>
1	e <sup>y</sup>
0	e <sup>y</sup>



$$\textcircled{6} \textcircled{a} \vec{c}(t) = \left( \frac{1}{2}t^2, \sqrt{2}t, \ln t \right) \quad \vec{F}(x, y, z) = \left( \frac{y}{\sqrt{2}}, \sqrt{2}, \frac{\sqrt{2}}{y} \right)$$

$$\vec{c}'(t) = \left( t, \sqrt{2}, \frac{1}{t} \right)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}\left(\frac{1}{2}t^2, \sqrt{2}t, \ln t\right) = \left(t, \sqrt{2}, \frac{1}{t}\right) = \vec{c}'(t) \quad \checkmark$$

$$\textcircled{b} \|\vec{c}'(t)\| = \sqrt{t^2 + 2 + \frac{1}{t^2}} = \sqrt{\left(t + \frac{1}{t}\right)^2} = t + \frac{1}{t}$$

$$\text{Length} = \int_1^2 \left(t + \frac{1}{t}\right) dt$$

$$= \left[ \frac{1}{2}t^2 + \ln t \right]_1^2$$

$$= (2 + \ln 2) - \left(\frac{1}{2} + 0\right)$$

$$= \frac{3}{2} + \ln 2$$