HW #3 — MTH254H

Evan's Key

$\boxed{1.3.6}$

Area of triangle $= \frac{1}{2}$ magnitude of cross product of side vectors

Side $\vec{v_1} = (1,1,1) - (0,0,0) = (1,1,1)$

$\vec{v_2} = (0,2,3)$

$$\left|\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}0\\-2\\3\end{pmatrix}\right| = \left|\begin{pmatrix}5\\-3\\-2\end{pmatrix}\right| = \sqrt{25+9+4} = \sqrt{38}$$

$$\underline{A = \frac{1}{2}\sqrt{38}}$$

$\boxed{1.3.8}$ *

Volume of parallelepiped = absolute value of determinant of the matrix w/ the 3 vectors as rows

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 4 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} + 0|\dots| + 0|\dots| = -1$$

$$\Rightarrow \boxed{1}$$

$\boxed{1.3.16}$ Use cross product to find a normal vector.

a. $\begin{pmatrix}2\\0\\-1\end{pmatrix} \times \begin{pmatrix}0\\4\\-3\end{pmatrix} = \begin{pmatrix}4\\6\\8\end{pmatrix} = \vec{n}$

$4(x-x_0) + 6(y-y_0) + 8(z-z_0) = 0$

$\underline{2x+3y+4z=0}$

b. $\vec{v_1} = (1-0, 2-1, 0-(-2)) = (1,1,2)$

$\vec{v_2} = (1-4, 2-0, 0-1) = (-3,2,-1)$

$\begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}-3\\2\\-1\end{pmatrix} = \begin{pmatrix}-5\\7\\5\end{pmatrix}$

$-5(x-1) + 7(y-2) + 5(z-0) = 0 \Rightarrow \underline{-5x+7y+5z=9}$

c. $\vec{v_1} = (2-0, -1-0, 3-5) = (2,-1,-2)$

$\vec{v_2} = (5-0, 7-0, -1-5) = (5,7,-6)$

Point $\vec{x_0} = (0,0,5)$.

$\begin{pmatrix}2\\-1\\-2\end{pmatrix} \times \begin{pmatrix}5\\7\\-6\end{pmatrix} = \begin{pmatrix}20\\2\\19\end{pmatrix}$

$20(x-0) + 2(y-0) + 19(z-5) = 0$

$\underline{20x + 2y + 19 = 95}$

$\boxed{1.3.19}$

(a.) If two planes are parallel, then their equations can be written like

P_1: $Ax + By + Cz = D_1$

P_2: $\alpha Ax + \alpha By + \alpha Cz = D_2$ since their normal vectors will be parallel.

If they're the same plane, then for a vector $\vec{v}$ in the plane, $(A, B, C) \cdot \vec{v} = D_1$ and $(\alpha A, \alpha B, \alpha C) \cdot \vec{v} = D_2$ and so $D_2 = \alpha D_1$, and they're the same plane, since we can divide both sides by α. If not true, then a vector on one plane isn't in the other, and they're never intersecting.

(b.) Two planes intersect in the form of a line, if the planes aren't parallel. You can see this by plugging one plane into another to "solve" the system of eqs and get an equation of a line in terms of one of the variables — e.g. $(6z+2, z-1, z)$.

$\boxed{1.3.22}$

Find the intersection of the two planes: $3(x-1) + 2y + (z+1) = 0$ and $(x-1) + 4y - (z+1) = 0$.

First, solve the second plane for x and plug into the first eq:

$x = 2 + z - 4y \Rightarrow 3(2+z-4y) - 3 + 2y + z + 1 = 0$

$\Rightarrow 6 + 3z - 12y - 2 + 2y + z = 0 \Rightarrow$ solve for, let's say, z.

$-10y + 4z + 4 = 0 \Rightarrow z = -1 + \frac{5}{2}y$. Now plug back in to an original plane EQ to find x in terms of y:

$x = 2 + (-1 + \frac{5}{2}y) - 4y = 1 - 1.5y$

To make it even clearer, we can set $y = t$ so it looks more like the parametrized functions we've worked with:

$\underline{\ell(t) = (1 - 1.5t, t, -1 + 2.5t)}$ (note: lots of answers work here!)

$\boxed{1.3.46}$

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ be orthogonal unit vectors; then for some basis, without loss of generality, we can say $\vec{v} = (1,0,0)$ and $\vec{w} = (0,1,0)$.

Then, $\vec{v} \times \vec{w} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \underset{=\vec{u}}{(0,0,1)}$ (which makes sense, cuz it has to be another unit vector)

So $\vec{u} \times \vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = (0,1,0) = \vec{w}$ ✓

$\vec{w} \times \vec{u} = \begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = (1,0,0) = \vec{v}$ ✓

$\boxed{1.4.3}$ R = rect, S = spherical

a. • $(1, 45°, 1) = (r, \theta, z)$
R: $(\sqrt{2}/2, \sqrt{2}/2, 1)$ S: $(\sqrt{2}, 45°, 45°)$

• $(2, \pi/2, -4) = (r, \theta, z)$
R: $(0, 2, -4)$ S: $(2\sqrt{5}, \pi/2, \approx 2.68 \approx 153° = \cos^{-1}(-4/2\sqrt{5}))$

• $(0, 45°, 10) = (r, \theta, z)$
R: $(0, 0, 10)$ S: $(10, 45°, 0°)$

• $(3, \pi/6, 4) = (r, \theta, z)$
R: $(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 4)$ S: $(5, \pi/6, \cos^{-1}(4/5) \approx 0.64 \text{ rad} \approx 36.9°)$

• $(2, 3\pi/4, -2) = (r, \theta, z)$
R: $(-\sqrt{2}, \sqrt{2}, -2)$ S: $(\sqrt{6}, 3\pi/4, \approx 2.53 \approx 145°)$

• $(1, \pi/6, 0) = (r, \theta, z)$
R: $(\sqrt{3}/2, 1/2, 0)$ S: $(3/4, \pi/6, \pi/2)$

flip

$\boxed{1.4.3}$

(b.) $(2,1,-2)=(x,y,z)$:

C: $(\sqrt{5}, \tan^{-1}(\frac{1}{2}) \approx 26.6°, -2)$ S: $(3, \tan^{-1}(\frac{1}{2}), \cos^{-1}(\frac{2}{3}) \approx 132°)$

$(0,3,4)=(x,y,z)$:

C: $(3, \pi/2, 4)$ S: $(5, \pi/2, \tan^{-1}(\frac{4}{5}) \approx 38.7°)$

$(\sqrt{2},1,1)=(x,y,z)$:

C: $(\sqrt{3}, \tan^{-1}(1/\sqrt{2}) \approx 35.3°, 1)$ S: $(2, \tan^{-1}(1/\sqrt{2}), \pi/3)$

$(-2\sqrt{3},-2,3)=(x,y,z)$:

C: $(4, \pi/6, 3)$ S: $(5, \pi/6, \cos^{-1}(\frac{3}{5}) \approx 53.1°)$

$\boxed{1.4.5}$

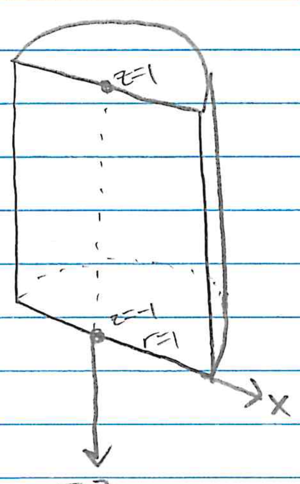
(a.) Rotate 180° "around the z-axis"

(b.) Reflect over the "x-y" plane

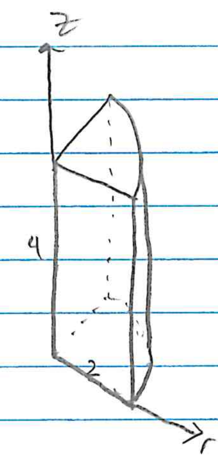
(c.) Double the distance from the origin (ie scale everything 2x) and rotate 90° "around the z-axis"

$\boxed{1.4.6}$ ✱

(a.) half a cylinder



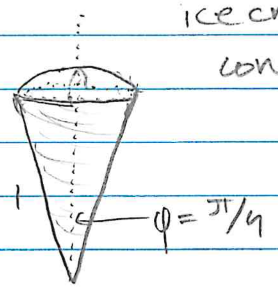
(b.)



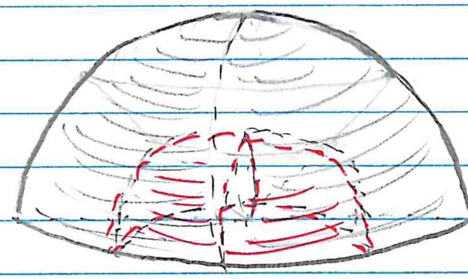
$\boxed{1.4.6 \text{ ctd}}$

Semi-hollow hemisphere

(c.) ice cream cone

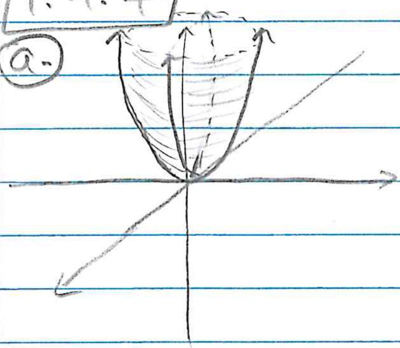


(d.)

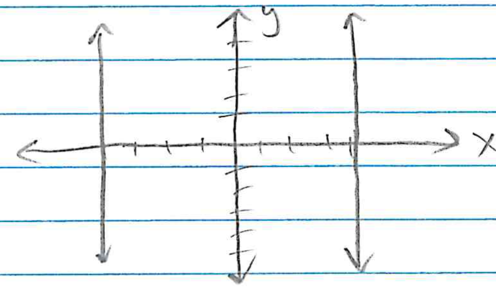


$\boxed{1.4.7}$

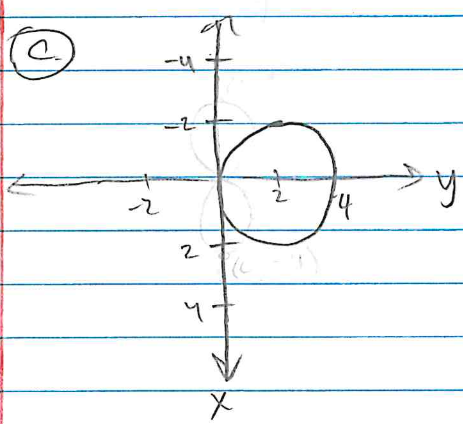
(a.)



(b.) Planes at $x=4$, $x=-4$. Look at the limits of each function part.



(c.)



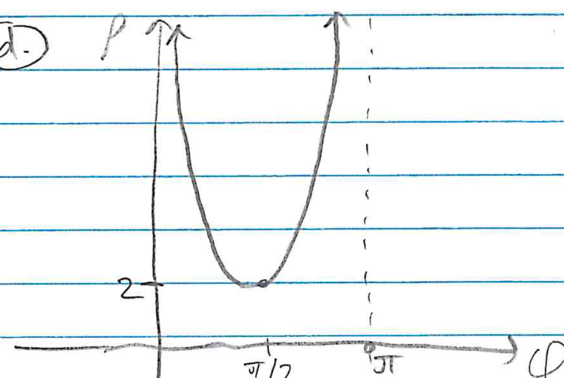
$r(r) = r(4\sin\theta)$

$r^2 = 4(r\sin\theta)$

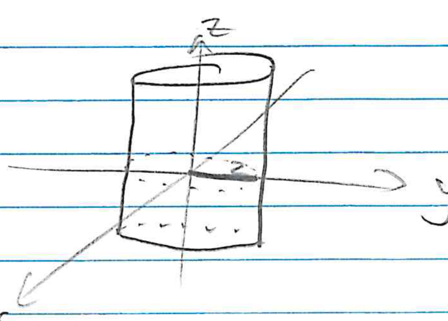
$x^2+y^2 = 4y$

$x^2+(y-2)^2 = 2^2$

(d.)



Making a tabb of points makes it clear that this defines a cylinder of radius 2 around the z-axis.



$\boxed{1.4.10}$

(a) Cylindrical: $1 \le r \le 3/2, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 8$

(b) Spherical: $4 \le \rho \le 6, \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi$

(c) Spherical: $0 \le \rho \le 5/2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi/2$

(d) Rectangular: $-1 \le x, y, z \le 1$

$\boxed{1.5.5}$

Cauchy-Schwarz: $\vec{x} \cdot \vec{y} = 5$

$$\|\vec{x}\| = \sqrt{5}, \quad \|\vec{y}\| = \sqrt{13}$$

$$5 = \sqrt{5} \cdot \sqrt{5} \le \sqrt{5} \cdot \sqrt{13} \checkmark$$

Triangle: $\|\vec{x} + \vec{y}\| = \|(4, -1, 1, -1, 3)\| = \sqrt{16+1+1+1+9} = 2\sqrt{7}$

so $\|\vec{x} + \vec{y}\| \approx 5.29$

$\|\vec{x}\| \approx 2.23 \qquad \Longrightarrow \quad 5.29 \le 2.23 + 3.61 \checkmark$

$\|\vec{y}\| = 3.61$

$\boxed{1.5.16}$

No. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad (A+B) = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\det(A) = 4, \quad \det(B) = 25, \quad \det(A+B) = 49.$

$4 + 25 \neq 49$, so it is not always true that $\det(A+B) = \det(A) + \det(B)$.

#3 *

(a.) $\{(1,2),(2,4)\}$ is <u>not</u>; $-2(1,2)+1(2,4)=\vec{0}$ = <u>dependent</u>

$\{(1,7),(2,0)\}$ <u>is</u>: $\alpha(1,7)+\beta(2,0)=(0,0) \Rightarrow \alpha=\beta=0$ = <u>independent</u>

(b.) Yes, $\{(2,0,1)\}$ is linearly <u>independent</u>.
Is $\{(1,0,4),(7,2,1),(0,2,0)\}$? Solve the system!

$$a(1,0,4)+b(7,2,1)+c(0,2,0)=(0,0,0)$$

$$\begin{cases} a+7b=0 \\ 2b+2c=0 \\ 4a+b=0 \end{cases}$$

Using rows 1, 3 we get $-27b=0$ so $b=0$; using row 2, we get $c=-b=0$; and so $a=0$ also, so this is <u>linearly independent</u>.

(c.) $\Rightarrow$: Assume $\vec{v}_1=(a,b)$, $\vec{v}_2=(c,d)\in\mathbb{R}^2$ are linearly dependent. Then $\exists\,\alpha\in\mathbb{R}$ such that:

$$\alpha\vec{v}_1+\vec{v}_2=\vec{0} \Rightarrow \alpha a+c=0,\ \alpha b+d=0$$

Solving for α, $\alpha=-c/a=-d/b$. Multiply each side by $-ab$ to get $bc=da$.

Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Its det is $ad-bc$, but those are equal $\Rightarrow$ the determinant is zero.

$\Leftarrow$: Assume $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix}=0$, so then $ad=bc$. Consider the vectors $\vec{v}_1=(a,b)$ and $\vec{v}_2=(c,d)$. They can be written as a linear combination $\alpha\vec{v}_1+\beta\vec{v}_2=(\alpha a+\beta c,\alpha b+\beta d)$.
Let $\alpha=d$ and $\beta=-b$. Then the vector $=(da-bc, da-ad)$
But $ad=bc$, so this $=\vec{0}$, and the two vectors must be linearly dependent.

$\boxed{\#3 \text{ ctd}}$

(d.) Assume $\{\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3), \vec{c}=(c_1,c_2,c_3)\}$ are linearly dependent in $\mathbb{R}^3$, so $\exists\, \alpha, \beta \in \mathbb{R}$ such that
$\vec{a}+\alpha\vec{b}+\beta\vec{c}=\vec{0}$. (why can we use only 2 scalars and not 3?)
Then $-a_1=\alpha b_1+\beta c_1$, $-a_2=\alpha b_2+\beta c_2$, and $-a_3=\alpha b_3+\beta c_3$.

Consider the det of
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}$$

$$= a_1(b_2c_3-b_3c_2) - a_2(b_1c_3-b_3c_1) + a_3(b_1c_2-b_2c_1)$$

Substitute in for a_i:

$$=(-\alpha b_1-\beta c_1)(b_2c_3-b_3c_2)-(-\alpha b_2-\beta c_2)(b_1c_3-b_3c_1)+(-\alpha b_3-\beta c_3)(b_1c_2-b_2c_1)$$

$$= \alpha(-b_1b_2c_3+b_1b_3c_2+b_1b_2c_3-b_2b_3c_1-b_1b_3c_2+b_2b_3c_1)$$
$$+\beta(-c_1c_3b_2+c_1c_2b_3+c_2c_3b_1-c_1c_2b_3-c_2c_3b_1+c_1c_3b_2)$$

$= \alpha(0)+\beta(0)$ since the terms all cancel

$= 0$, woo!

Alternatively: use the geometry of the determinant. Linearly independent vectors in $\mathbb{R}^3$ lie in the same plane, which means one of the dimensions of the parallelepiped it spans is zero. Determinant measures the volume of that parallelepiped, which must be zero since one of its dimensions is zero!

#4

(a) Recall: $R_\theta\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$, $F\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Let $\vec{x}=(a,b)$ and $\vec{y}=(c,d)$. Start with rotation:

$R_\theta(\vec{x})\cdot R_\theta(\vec{y}) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a\cos\theta - b\sin\theta\\ a\sin\theta + b\cos\theta\end{bmatrix}\cdot\begin{bmatrix}c\cos\theta - d\sin\theta\\ c\sin\theta + d\cos\theta\end{bmatrix} = ac\cos^2\theta - (bc+ad)\cos\theta\sin\theta + bd\sin^2\theta$
$+ ac\sin^2\theta + (bc+ad)\cos\theta\sin\theta + bd\cos^2\theta$

$= ac(\cos^2\theta+\sin^2\theta) + bd(\cos^2\theta+\sin^2\theta)$

$= ac+bd = \begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix} = \vec{x}\cdot\vec{y}$ ✓

Consider $F(\vec{x})$: $F\begin{pmatrix}a\\b\end{pmatrix}\cdot F\begin{pmatrix}c\\d\end{pmatrix} = \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}\right) = \begin{pmatrix}a\\-b\end{pmatrix}\cdot\begin{pmatrix}c\\-d\end{pmatrix}$

$= ac+bd = \begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix} = \vec{x}\cdot\vec{y}$ ✓

(b) If $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c_1\\d_1\end{pmatrix}$ and $f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_2\\d_2\end{pmatrix}$, then the matrix of the transformation is $f(\vec{x}) = \begin{bmatrix}c_1 & c_2\\ d_1 & d_2\end{bmatrix}\vec{x}$. We know this preserves lengths, so $|\vec{e_1}| = |f(\vec{e_1})|$, and $|\vec{e_2}| = |f(\vec{e_2})|$. That means $1 = \sqrt{c_1^2+d_1^2} = c_1^2+d_1^2$, and $1 = \sqrt{c_2^2+d_2^2} = c_2^2+d_2^2$. (This can be equivalently written as $f(\vec{e_1})\cdot f(\vec{e_1}) = f(\vec{e_2})\cdot f(\vec{e_2}) = 1$.)

Finally, we know the isometry preserves angles and $\vec{e_1}\perp\vec{e_2}$, so $f(\vec{e_1})\perp f(\vec{e_2})$ also, and $\begin{pmatrix}c_1\\d_1\end{pmatrix}\cdot\begin{pmatrix}c_2\\d_2\end{pmatrix} = c_1c_2 + d_1d_2 = 0$

What does this all mean? Well, $1 = \sqrt{c_1^2+d_1^2} = \sqrt{c_2^2+d_2^2}$ are the equations for two-dimensional unit circles, so this means (c_1,d_1) and (c_2,d_2) must be two vectors on the unit circle, AND they must be perpendicular to each other. That's it!

It can be shown that this only encompasses rotations, reflections, and their products, since we can prove if $f(\vec{x}), g(\vec{x})$ are isometries then $(f\circ g)(\vec{x})$ is also an isometry, so we can generalize this to a really interesting concept about isometries in $\mathbb{R}^2$ being such products!