

1.2.13

$$* (5, b, c) \cdot (1, 2, 3) = 0$$

$$5 + 2b + 3c = 0$$

$$(5 + 2b + 3c) + (5 - 2b + c) = 0 + 0$$

$$10 + 4c = 0$$

$$c = -\frac{5}{2}$$

$$\rightarrow 5 - 2b - \frac{5}{2} = 0$$

$$-2b = -2.5 \rightarrow b = 1.25 = \frac{5}{4}$$

$$\Rightarrow (5, \frac{5}{4}, -\frac{5}{2})$$

1.2.16

$$A) \vec{u} = (15, 2, 4)$$

$$|\vec{u}| = \sqrt{225 + 4 + 16} = \sqrt{245} = 7\sqrt{5}$$

$$\text{Normalized} = \frac{1}{7\sqrt{5}} \vec{u} = \left(\frac{15}{7\sqrt{5}}, \frac{2}{7\sqrt{5}}, \frac{4}{7\sqrt{5}} \right)$$

$$= \left(\frac{3\sqrt{5}}{7}, \frac{-2\sqrt{5}}{35}, \frac{4\sqrt{5}}{35} \right)$$

$$\vec{v} = (7, 3, -1)$$

$$|\vec{v}| = \sqrt{49 + 9 + 1} = \sqrt{10 + 49} = \sqrt{59}$$

$$\text{Normalized} = \frac{1}{\sqrt{59}} \vec{v}$$

$$= \left(\frac{7}{\sqrt{10+49}}, \frac{3}{\sqrt{10+49}}, \frac{-1}{\sqrt{10+49}} \right)$$

$$B) \vec{u} = (-1, 2, 0)$$

$$|\vec{u}| = \sqrt{1 + 4} = \sqrt{5}$$

$$\text{Normalized} = \frac{1}{\sqrt{5}} \vec{u} = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right)$$

$$\vec{v} = (1, -1, 0)$$

$$|\vec{v}| = \sqrt{1 + 1 + 0} = \sqrt{2}$$

$$\text{Normalized} = \frac{1}{\sqrt{2}} \vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right)$$

$$= \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right)$$

$$C) \vec{u} = (5, -1, 2)$$

$$|\vec{u}| = \sqrt{25 + 1 + 4} = \sqrt{30}$$

$$\text{Normalized} = \frac{1}{\sqrt{30}} \vec{u} = \left(\frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{2}{\sqrt{30}} \right)$$

$$\vec{v} = (-2, -3, -7)$$

$$|\vec{v}| = \sqrt{4 + 9 + 49} = \sqrt{62}$$

$$\text{Normalized} = \left(\frac{-2}{\sqrt{62}}, \frac{-3}{\sqrt{62}}, \frac{-7}{\sqrt{62}} \right)$$

Notice a mistake? Be the first one to email me and get a free candy bar! (or some equivalent prize)

1.2.19Orthogonal \Rightarrow their dot product is zero

$$(7, x, -10) \cdot (3, x, x) = 21 + x^2 - 10x = 0$$

This is a quadratic:

$$x^2 - 10x + 21 = (x-7)(x-3) = 0$$

 $\Rightarrow x=7, 3$ are the only solutions.1.2.25

Two nonparallel vectors orthogonal to $\vec{x} = (1, 1, 1)$: their respective dot products are zero. Let $\vec{v}_1 = (a, b, c)$ and $\vec{v}_2 = (d, e, f)$ with

$$\vec{v}_2 \cdot \vec{x} = 0 \Rightarrow d + e + f = 0$$

$$\vec{v}_1 \cdot \vec{x} = 0 \Rightarrow a + b + c = 0$$

There are many different valid solutions to this problem! You can start with any two a, b and d, e and then solve the system to find c and f . Two examples are $(0, 4, -4)$ and $(1, 2, -3)$. The only caveat is that you must prove the two vectors you chose are nonparallel: i.e. there does not exist $\alpha \in \mathbb{R}$ such that $\alpha \vec{v}_1 = \vec{v}_2$, which is simple to do by showing the resulting system has no solutions: for example:

Assume $\exists \alpha \in \mathbb{R}$ such that $\alpha \vec{v}_1 = \vec{v}_2$. Then

$\alpha(0, 4, -4) = (1, 2, -3)$ and $0=1$, which is a contradiction, so our premise was false and no such scalar exists.

1.2.26

Find the line through $(3, 1, -2)$ that intersects and is perpendicular to the line $\ell_1(t) = (-1+t, -2+t, -1+t)$.

Our line will be of the form $\ell_2(t) = (3, 1, -2) + t(a, b, c)$ where $(a, b, c) \cdot (1, 1, 1) = 0$ (perpendicular) and at time s , both lines meet at $(x_0, y_0, z_0) = \vec{p}_0$:

$$\ell_1(s) = \vec{p}_0 \Rightarrow (x_0, y_0, z_0) = (-1+s, -2+s, -1+s)$$

$$\ell_2(s) = \vec{p}_0 \Rightarrow (x_0, y_0, z_0) = (3+sa, 1+sb, -2+sc)$$

We can then solve for a, b, c : $a = -\frac{4+s}{s}$, $b = -\frac{3+s}{s}$, $c = \frac{1+s}{s}$.

But by \perp , $a+b+c=0$, so we can solve for s :

$$\left(-\frac{4+s}{s}\right) + \left(-\frac{3+s}{s}\right) + \left(\frac{1+s}{s}\right) = 0$$

$$-6+3s=0 \Rightarrow s=2$$

So by plugging back into $\ell_1(2)$, we can find the intersect pt:

$$(x_0, y_0, z_0) = (1, 0, 1)$$

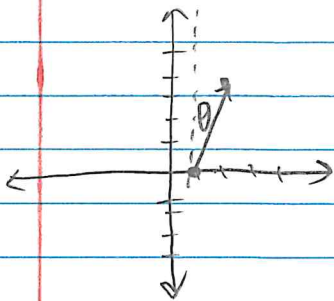
From there we can solve for a, b, c : $\ell_2(2) = (1, 0, 1)$

$$= (3+2a, 1+2b, -2+2c)$$

$$\Rightarrow a = -1, b = -\frac{1}{2}, \text{ and } c = \frac{3}{2}.$$

$$\Rightarrow \underline{\ell_2(t) = (3, 1, -2) + t(-1, -\frac{1}{2}, \frac{3}{2})}$$

1.2.29



a) $(2, 4) - (1, 0) = (1, 4) = \vec{v}$

b) $\vec{v} \cdot (0, 1) = |\vec{v}| |(0, 1)| \cos \theta$

$1 \cdot 0 + 4 \cdot 1 = \sqrt{1^2 + 4^2} \cdot \sqrt{1} \cos \theta$

$\theta = \arccos(4/\sqrt{17}) \approx 0.245 \text{ rad}$
 $\approx 14.04^\circ$

1.2.31

$\ell(t) = (3, 4, 5) + t(400, 500, -1)$ equivalent to $x(t) = x_0 + vt$

where is it when $x = 23, y = 29$?

$3 + 400t = 23 \Rightarrow t = 0.05$

$4 + 500t = 29 \Rightarrow t = 0.05 \checkmark$

so (a) $t = 0.05 \text{ hr (3 min)}$
 $= 12:03 \text{ pm}$

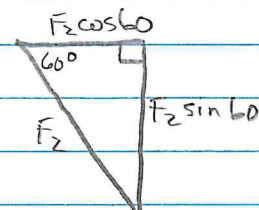
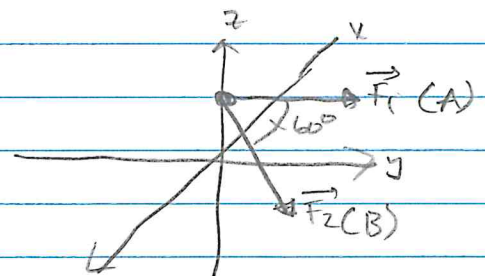
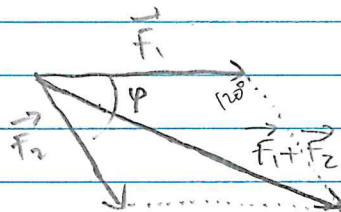
(b) what is its height at this time?

$z(t) = 5 - t = 4.95 \text{ km}$

1.2.34

a) $|\vec{F}_1| = 150$

$|\vec{F}_2| = 110$



b) $\vec{F}_1 = (150, 0)$

$\vec{F}_2 = (110 \cos 60, 110 \sin 60)$

$= (55, 55\sqrt{3})$

Finding ϕ : $\vec{F}_1 \cdot (\vec{F}_1 + \vec{F}_2) = |\vec{F}_1| |\vec{F}_1 + \vec{F}_2| \cos \phi$

$30750 = (150)(10\sqrt{511}) \cos \phi$

$\phi = \arccos(20.5/\sqrt{511}) \approx 0.435 \text{ rad} \approx 24.9^\circ$

1.3.7

Volume of parallelepiped = determinant of matrix with vectors as its rows: $(2, 1, -1)$, $(5, 0, -3)$, and $(1, -2, 1)$ go to:

$$\begin{vmatrix} 2 & 1 & -1 \\ 5 & 0 & -3 \\ 1 & -2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & -3 \\ -2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 5 & -3 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 5 & 0 \\ 1 & -2 \end{vmatrix} \\ = 2(-6) - 1(8) + 10 = -10 \text{ units}^3$$

1.3.9

All unit vectors perpendicular to $(1, 1, 0) = \sum (a, b, c) \mid a+b=0, \sqrt{a^2+b^2+c^2}=1$
 1) Obvious: $\vec{k} = (0, 0, 1)$.
 2) Less obvious: $\vec{v} = (r, -r, \sqrt{1-2r^2})$ for $0 \leq r \leq \frac{\sqrt{2}}{2}$

Finding vectors perpendicular to \vec{a}, \vec{b} : they're on the line $\vec{a} \times \vec{b}$!

$\vec{c} \times \vec{d} = \vec{k}$, so all unit vectors orthogonal to \vec{c}, \vec{d} are $\pm \vec{k}$.

1.3.10

Same idea as above. $\vec{x} = (-5, 9, 4)$, $\vec{y} = (7, 8, 9)$:

$$\vec{x} \times \vec{y} = \begin{pmatrix} -5 \\ 9 \\ 4 \end{pmatrix} \times \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = (49, 73, -103) = \vec{z}$$

Normalize: $|\vec{z}| \approx 135.4$ (exactly $\sqrt{18339}$)

\Rightarrow Vectors orthogonal to \vec{x}, \vec{y} are of the form

$$\frac{\pm 1}{\sqrt{18339}} (49, 73, -103)$$

1.3.13 $\vec{u} = (1, -2, 1)$ $\vec{v} = (2, -1, 2)$

$$\vec{u} + \vec{v} = (3, -3, 3)$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + (-2)(-1) + 1 \cdot 2 = 6$$

$$\|\vec{u}\| = \sqrt{6}$$

$$\|\vec{v}\| = 3$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = (-3, 0, 3)$$

* **1.3.15**

$$\vec{n} = (A, B, C)$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

for planes!

Find eq of the plane that

(a) is perpendicular to $\vec{n} = (1, 1, 1)$ and passes thru $(1, 0, 0)$

$$1(x-1) + 1(y-0) + 1(z-0) = 0$$

$$\underline{x + y + z = 1}$$

(b) perpendicular to $\vec{n} = (1, 2, 3)$, passes thru $(1, 1, 1)$

$$1(x-1) + 2(y-1) + 3(z-1) = 0$$

$$\underline{x + 2y + 3z = 6}$$

(c) perpendicular to $\vec{n} = (5, 0, 2)$ (we can just extract this from the line eq) and passes thru $(5, -1, 0)$

$$5(x-5) + 0(y+1) + 2(z-0) = 0$$

$$\underline{5x + 2z = 25}$$

(d) $\vec{n} = (-1, -2, 3)$ thru $(2, 4, -1)$

$$-1(x-2) - 2(y-4) + 3(z+1) = 0$$

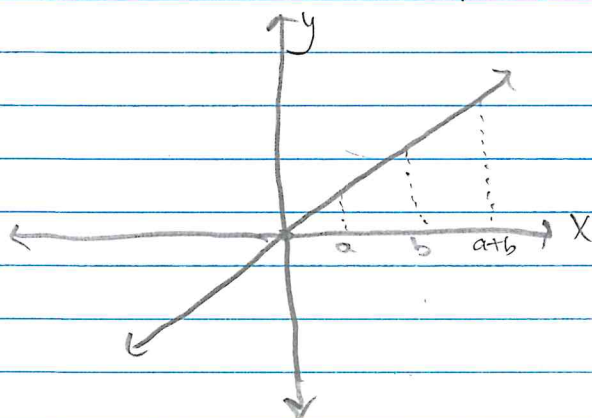
$$\underline{-x - 2y + 3z = -13}$$

Question #4

(a.) A line $l(t) = \vec{a} + t\vec{v}$, $l: \mathbb{R} \rightarrow \mathbb{R}^3$, is a linear transformation precisely when $\vec{a} = \vec{0}$, i.e. the line begins at the origin. We can show this using the definition of a linear transformation:

$$\begin{aligned} l(s+t) &= l(s) + l(t) \\ \vec{a} + (s+t)\vec{v} &= (\vec{a} + s\vec{v}) + (\vec{a} + t\vec{v}) \\ \vec{a} + s\vec{v} + t\vec{v} &= 2\vec{a} + s\vec{v} + t\vec{v} \Rightarrow \vec{a} = 2\vec{a}, \text{ so } \vec{a} = \vec{0}. \end{aligned}$$

This makes sense because linear transformations just scalar multiply or rotate vectors; they can't translate ("slide") them. Think about $\mathbb{R}^1 \rightarrow \mathbb{R}^1$, i.e. $y = mx + b$:



we only have $y(a) + y(b) = y(a+b)$ when the y -intercept $= 0$.

(b.) Think about any vector $\vec{v} = (x, y) \in \mathbb{R}^2$. We can write this as a linear combination of the basis vectors: $\vec{v} = x\vec{e}_1 + y\vec{e}_2$. That's cool because we can now harness the properties of linearity to figure out all of $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if we just know $F(\vec{e}_1)$ and $F(\vec{e}_2)$:

$$\begin{aligned} \text{For arbitrary } \vec{v}, \quad F(\vec{v}) &= F(x\vec{e}_1 + y\vec{e}_2) = F(x\vec{e}_1) + F(y\vec{e}_2) \\ &= xF(\vec{e}_1) + yF(\vec{e}_2) \quad !!! \end{aligned}$$

Let's say $F(\vec{e}_1) = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$ and $F(\vec{e}_2) = \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$. Then, we can write: (next ps)

Question #4 ctd ...

$$\vec{v} = x f(\vec{i}) + y f(\vec{j}) = x \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} + y \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} xc_1 + yc_2 \\ xd_1 + yd_2 \end{pmatrix}$$

Cool. How does that become a matrix? Well, let's reverse engineer it: what does a matrix product look like?

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ta + ub \\ va + wb \end{bmatrix}$$

From here, we can see that $c_1 \leftrightarrow t$, $c_2 \leftrightarrow u$, $d_1 \leftrightarrow v$, and $d_2 \leftrightarrow w$,

so we can write:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

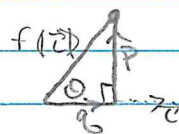
and, if you look closely, you'll notice the matrix is really just $[f(\vec{i}) | f(\vec{j})]$

(C). Rotation counterclockwise by θ is a linear transformation with

$$R_\theta(\vec{v}) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{v}$$

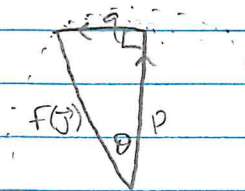
How do we know

that? Look at how it treats \vec{i}, \vec{j} :



$$\begin{aligned} p &= \sin \theta |\vec{i}| \\ q &= \cos \theta |\vec{i}| \end{aligned}$$

$$\Rightarrow (1, 0) \mapsto (\cos \theta, \sin \theta)$$



$$p = \cos \theta |\vec{j}|$$

$$q = -\sin \theta |\vec{j}| \Rightarrow (0, 1) \mapsto (-\sin \theta, \cos \theta)$$

In the same way, look at how reflection over the x axis affects $(1, 0)$ and $(0, 1)$:

$$(1, 0) \mapsto (1, 0)$$

$$(0, 1) \mapsto (0, -1)$$

$$\text{so } F(\vec{v}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{v}$$

$$= [f(\vec{i}) | f(\vec{j})]$$