**5.2.1**

(a.) $\int_0^1 \int_0^1 x^3 + y^2 \, dx \, dy = \int_0^1 \frac{1}{4} + y^2 \, dy = \boxed{7/12}$

(b.) $\int_0^1 \int_0^1 y e^{xy} \, dx \, dy = \int_0^1 \left[ e^{xy} \Big|_0^1 \right] dy = \int_0^1 e^y - 1 \, dy = \underline{e-2}$

(c.) $\int_0^1 \int_0^1 x^2 y^2 \cos(x^3) \, dx \, dy = \int_0^1 \left[ \frac{1}{3} y^2 \sin(x^3) \Big|_0^1 \right] dy = \int_0^1 \frac{1}{3} y^2 \sin(1) \, dy$

$= \underline{\frac{1}{9} \sin(1)}$

(d.) $\int_0^1 \int_0^1 \ln \left[ (x+1)(y+1) \right] dy \, dx \Rightarrow$ use integration by parts to get

$= \int_0^1 \left[ (y+1) \ln[(x+1)(y+1)] - y + \ln(y+1) \Big|_0^1 \right] dx$   OR it's easier to use...

~~$= \int_0^1 [2\ln[2x+2] + \ln[2] - \ln[(x+1)] \, dx$~~

~~$= [(2x+2)\ln[2x+2] - 2x + x + x\ln[2] - (x+1)\ln[x+1] + x] \Big|_0^1$~~

~~$= [(4\ln(4) - 2 + \ln(2) + 2\ln(2)) - (2\ln(2)) + \ln(4) + \ln(4) - \ln(2) - 2]$~~

~~$= 2\ln 4 + \ln 2$~~

$= \int_0^1 \int_0^1 \ln(x+1) + \ln(y+1) \, dy \, dx = \int_0^1 y\ln(x+1) + (y+1)\ln(y+1) - y \Big|_0^1 \, dx$

$= \int_0^1 \ln(x+1) + 2\ln(2) - 1 \, dx = \left[ (x+1)\ln(x+1) - x + 2x\ln 2 - x \right] \Big|_0^1$

$= 2\ln(2) - 1 + 2\ln(2) - 1 = \underline{4\ln(2) - 2}$

5.2.3

$$\int_{0}^{2} \int_{-1}^{1} \frac{yx^3}{y^2+2} \, dy \, dx = \int_{-1}^{1} \int_{0}^{2} \frac{yx^3}{y^2+2} \, dx \, dy = \int_{-1}^{1} \left[ \frac{yx^4}{4y^2+8} \right]_{0}^{2} dy$$

$$= \int_{-1}^{1} \frac{16y}{4y^2+8} \, dy = \int_{-1}^{1} \frac{4y}{y^2+2} = \left[ 2\ln(y^2+2) \right]_{-1}^{1}$$

$$= 2\ln(3) - 2\ln(3) = \underline{0}$$

5.2.8

Bounds: $y=0, \; x=0, \; z=0, \; x=1, \; y=1, \; z=x^2+y^4$

$$\implies \int_{0}^{1} \int_{0}^{1} x^2+y^4 \, dx \, dy = \int_{0}^{1} \left[ \frac{1}{3}x^3 + xy^4 \right]_{0}^{1} dy = \int_{0}^{1} \frac{1}{3}+y^4 \, dy$$

$$= \left[ \frac{1}{3}y + \frac{1}{5}y^5 \right]_{0}^{1} = \frac{1}{3} + \frac{1}{5} = \underline{8/15}$$

5.3.1

a) iii

b) iv

c) ii
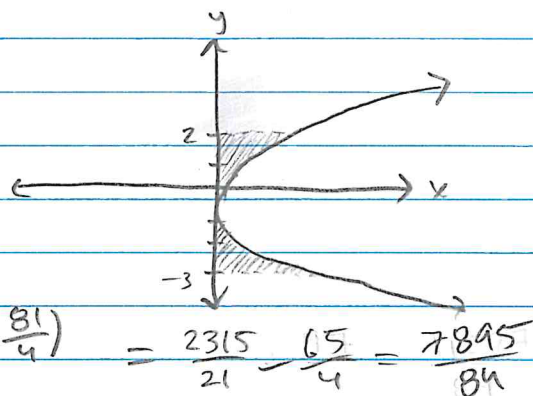
d) i

5.3.4

a.) $\int_{-3}^{2} \int_{0}^{y^2} (x^2+y)\, dx\, dy$

$= \int_{-3}^{2} \left[ \frac{1}{3}x^3 + xy \Big|_0^{y^2} \right] dy = \int_{-3}^{2} \frac{1}{3}y^6 + y^3$

$= \left[ \frac{1}{21}y^7 + \frac{1}{4}y^4 \right]_{-3}^{2} = \left( \frac{128}{21}+4 \right) - \left( \frac{-2187}{21} + \frac{81}{4} \right) = \frac{2315}{21} - \frac{65}{4} = \frac{7895}{84}$
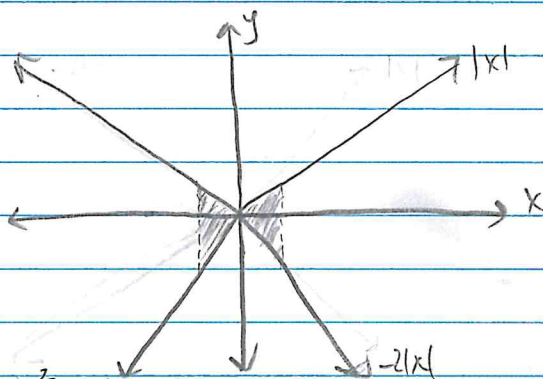
$\approx 94$

b.) $\int_{-1}^{1} \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$

$= \int_{-1}^{0} \int_{2x}^{-x} e^{x+y}\, dy\, dx + \int_0^1 \int_{-2x}^{x} e^{x+y}\, dy\, dx$
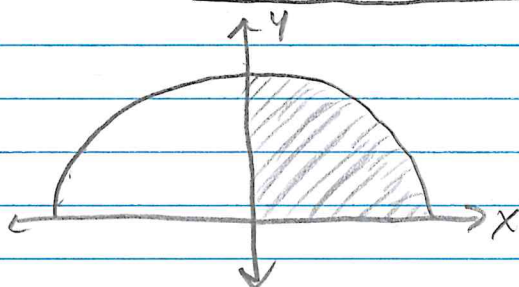
$= \int_{-1}^{0} 1 - e^{3x}\, dx + \int_0^1 e^{2x} - e^{-x}\, dx$

$= \left[ x - \frac{1}{3}e^{3x} \right]_{-1}^{0} + \left[ \frac{1}{2}e^{2x} + e^{-x} \right]_0^1 = -\frac{1}{3} + 1 + \frac{1}{3e^3} + \frac{e^2}{2} + \frac{1}{e} - \frac{1}{2} - 1$

$= \frac{1}{3}e^{-3} + \frac{1}{2}e^2 + e^{-1} - \frac{5}{6}$

c.) $\int_0^1 \int_0^{\sqrt{1-x^2}} 1\, dy\, dx = \int_0^1 \sqrt{1-x^2}\, dx$
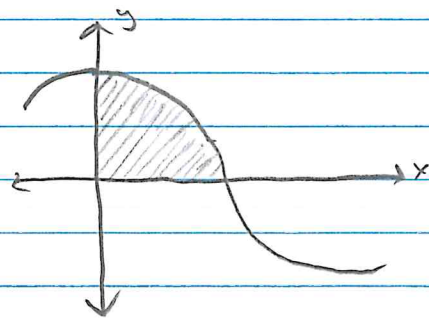
$= \frac{\pi}{4}$   (it's a quarter of the unit circle!)

d.) $\int_0^{\pi/2} \int_0^{\cos x} y\sin(x)\, dy\, dx$

$= \int_0^{\pi/2} \left[ \frac{1}{2}y^2 \sin(x) \Big|_0^{\cos x} \right] dx = \int_0^{\pi/2} \frac{1}{2}\cos^2 x \sin x\, dx$

$= \frac{1}{2} \left[ -\frac{1}{3}\cos^3 x \Big|_0^{\pi/2} \right] = \frac{1}{2} \left[ 0 - \left( -\frac{1}{3} \right) \right] = \frac{1}{6}$

5.3.4 ctd

e) $\int_0^1 \int_{y^2}^4 (x^n + y^m) \, dx \, dy$

$= \int_0^1 \left[ \frac{1}{n+1} x^{n+1} + x y^m \right]_{y^2}^4 dy = \int_0^1 \left[ \frac{1}{n+1} y^{n+1} + y^{m+1} - \frac{1}{n+1} y^{2n+2} - y^{m+2} \right] dy$
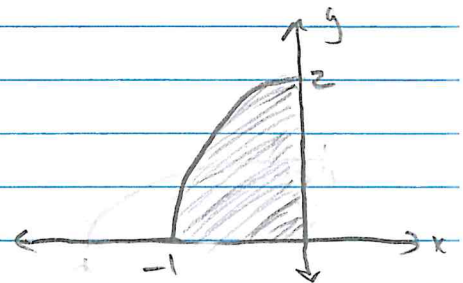
$= \left[ \frac{y^{n+2}}{(n+1)(n+2)} + \frac{y^{m+2}}{m+2} - \frac{y^{2n+3}}{(n+1)(2m+3)} - \frac{y^{m+3}}{(m+3)} \right]_0^1$

$= \underline{\frac{1}{(n+1)(n+2)} + \frac{1}{(m+2)} - \frac{1}{(n+1)(2m+3)} - \frac{1}{(m+3)}}$

f) $\int_{-1}^0 \int_0^{2\sqrt{1-x^2}} x \, dy \, dx$

$= \int_{-1}^0 \left[ xy \right]_0^{2\sqrt{1-x^2}} dx = \int_{-1}^0 2x\sqrt{1-x^2} \, dx$

$= \left[ -\frac{2}{3}(1-x^2)^{3/2} \right]_{-1}^0 = -\frac{2}{3} + \frac{2}{3}(0)^{3/2} = \underline{-\frac{2}{3}}$



$\boxed{5.3.8}$

Bounds: $x=0, \, y=0, \, 3x+4y=10$, i.e. $x = \frac{10}{3}$ and $y = \frac{10-3x}{4}$

$\iint x^2+y^2 \, dA = \int_0^{10/3} \int_0^{10-3x/4} x^2+y^2 \, dy \, dx = \int_0^{10/3} \left[ yx^2 + \frac{1}{3}y^3 \Big|_0^{\frac{10-3x}{4}} \right] dx$

$= \int_0^{10/3} \frac{1}{4}(10x^2 - 3x^3) + \frac{1}{3}\left( \frac{10-3x}{4} \right)^3 dx$

$= \int_0^{10/3} \frac{10}{4}x^2 - \frac{3}{4}x^3 - \frac{9}{64}x^3 + \frac{45}{32}x^2 - \frac{75}{16}x + \frac{125}{24} \, dx$

$= \left[ \frac{10}{12}x^3 - \frac{3}{16}x^4 - \frac{9}{256}x^4 + \frac{45}{96}x^3 - \frac{75}{32}x^2 + \frac{125}{24}x \right]_0^{10/3}$

$= \frac{10000}{324} - \frac{30000}{1296} - \frac{90000}{20736} + \frac{45000}{2592} - \frac{7500}{288} + \frac{1250}{72} = \underline{\frac{15625}{1296}} \approx 12$

**5.3.9**

Bounds given: $x=0$, $x=-4y^2+3$

Find bounds of $y$ from the roots of $x(y)$: $y=\pm\frac{\sqrt{3}}{2}$

$$\Rightarrow \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_0^{-4y^2+3} x^3y\,dx\,dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\left[\frac{1}{4}x^4y\right]\Big|_0^{-4y^2+3}dy$$

$$=\int_{-\sqrt{3}/2}^{\sqrt{3}/2} 64y^9-192y^7+216y^5-108y^3+\frac{81}{4}y\,dy$$

$$=\left[6.4y^{10}-24y^8+36y^6-27y^4+\frac{81}{8}y^2\right]\Big|_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

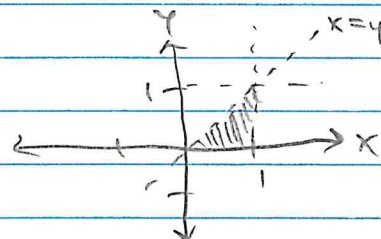This is an even function $\Rightarrow$ the answer is **zero**

**5.4.2**

Bounds given: $0\leq y\leq 1$ and $y\leq x\leq 1$,
which looks like

So we can write $0\leq x\leq 1,\ 0\leq y\leq x$:

$$\int_0^1\int_0^x \sin(x^2)\,dy\,dx = \int_0^1 x\sin x^2\,dx$$

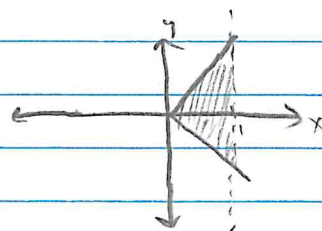$$=\left[-\frac{1}{2}\cos(x^2)\right]\Big|_0^1 = -\frac{1}{2}\cos(1)+\frac{1}{2} = \frac{1}{2}[1-\cos(1)]$$



**5.4.4**

a) $\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2\,dx\,dy \Rightarrow 0\leq x\leq 1,\ -x\leq y\leq x$:

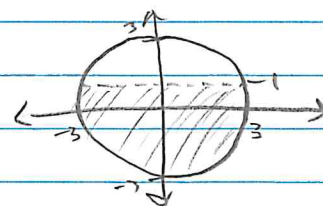$$\int_0^1\int_{-x}^x x^2+y^2+2xy\,dy\,dx = \int_0^1\left[yx^2+\frac{1}{3}y^3+xy^2\right]\Big|_{-x}^x dx$$

$$=\int_0^1 (x^3+\frac{1}{3}x^3+x^3)-(-x^3-\frac{1}{3}x^3+x^3)\,dx = \int_0^1 \frac{8}{3}x^3\,dx = \left[\frac{8}{12}x^4\right]\Big|_0^1 = \underline{8/12}$$



b) $-\sqrt{9-y^2}\leq x\leq \sqrt{9-y^2}$, $-3\leq y\leq 1$

$$\int_{-3}^1\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2\,dx\,dy = \int_{-3}^1\left[\frac{1}{3}x^3\Big|_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\right]dy$$

$$=\int_{-3}^1 \frac{2}{3}(9-y^2)^{3/2}\,dy \qquad ???$$

### 5.4.4 ctd

That didn't work.. so what now? Change order of integration, or even better, take the integral of the whole circle and subtract out the top part:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2\, dy\, dx - \int_{-2\sqrt{2}}^{2\sqrt{2}}\int_{1}^{\sqrt{9-x^2}} x^2\, dy\, dx$$

$$= 2\int_{-3}^{3} x^2\sqrt{9-x^2}\, dx - \int_{-2\sqrt{2}}^{2\sqrt{2}} x^2\left(\sqrt{9-x^2}-1\right) dx \quad \text{(See back of book, #52)}$$

$$= 2\left[\frac{x}{8}(2x^2-9)\sqrt{9-x^2} + \frac{81}{8}\arcsin\left(\frac{x}{3}\right)\right]_{-3}^{3} - \left[\frac{x}{8}(2x^2-9)\sqrt{9-x^2} + \frac{81}{8}\arcsin\left(\frac{x}{3}\right) - \frac{1}{3}(x^3)\Big|_{-2\sqrt{2}}^{2\sqrt{2}}\right]$$
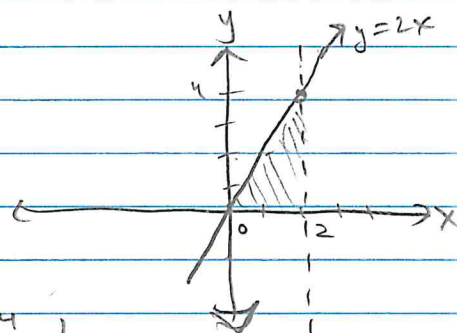
$$= 6\left[\frac{3}{8}(9)(0) + \frac{81}{8}\arcsin(1)\right] - \left[\left\{\frac{2\sqrt{2}}{8}(8-9)(1) + \frac{81}{8}\arcsin(\sqrt{2}/6) - \frac{16\sqrt{2}}{3}\right\} + \left\{\frac{-2\sqrt{2}}{8}(8-9)(1) + \frac{81}{8}\arcsin(-\sqrt{2}/6) + \frac{16\sqrt{2}}{3}\right\}\right]$$

$$= \frac{81\pi}{16} - \frac{\sqrt{2}}{2} - \frac{32\sqrt{3}}{2} = \frac{81\pi}{16} - \frac{33\sqrt{3}}{2}$$

c) $\int_{0}^{4}\int_{y/2}^{2} e^{x^2}\, dx\, dy$
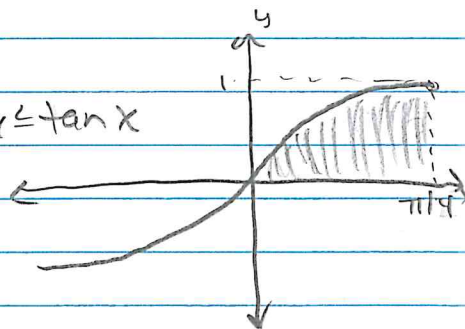


Put $0 \le x \le 2$, $0 \le y \le 2x$

$$\int_{0}^{2}\int_{0}^{2x} e^{x^2}\, dy\, dx = \int_{0}^{2} 2x e^{x^2}\, dx = \left[e^{x^2}\Big|_{0}^{2}\right] = e^4 - 1$$

d) $\int_{0}^{1}\int_{\tan^{-1}y}^{\pi/4} \sec^5 x\, dx\, dy$. Set $0 \le x \le \frac{\pi}{2}$, $0 \le y \le \tan x$



$$= \int_{0}^{\pi/4}\int_{0}^{\tan x} \sec^5 x\, dy\, dx = \int_{0}^{\pi/4} \tan x \sec^5 x\, dx$$

$$= \left[\frac{1}{5}\sec^5(x)\Big|_{0}^{\pi/4}\right] = \frac{1}{5}(\sqrt{2}^5 - 1^5) = \frac{4\sqrt{2}-1}{5}$$

**5.4.6**

a.) $orig = \int_0^{\pi/4} \cos x - \sin x \, dx = [\sin x + \cos x \big|_0^{\pi/4} = \sqrt{2} - 1$

Second version: $\int_0^{\sqrt{2}/2} \arcsin y \, dy + \int_{-\sqrt{2}/2}^{2} \arccos y \, dy$

$= [y \arcsin y + \sqrt{1-y^2} \big|_0^{\sqrt{2}/2} + [y \arccos y - \sqrt{1-y^2} \big|_{\sqrt{2}/2}^{2}$

$= \frac{\sqrt{2}}{2}(\frac{\pi}{4}) + \frac{1}{4} - 1 + 2\arccos 2 - i - \frac{\sqrt{2}}{2}(\frac{\pi}{4}) + \frac{1}{4}$     <u>FALSE</u>

b) TRUE

c) TRUE

d) TRUE

## 5.4.15

$$\iint_D y^3(x^2+y^2)^{-3/2}\,dxdy \qquad D: \frac{1}{2} \leq y \leq 1$$
$$x^2+y^2 \leq 1$$

D is x-simple:

$x: \left[-\sqrt{3}/2,\ \sqrt{3}/2\right]$

$y: \left[1/2,\ \sqrt{1-x^2}\right]$



$$\Rightarrow \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{1/2}^{\sqrt{1-x^2}} y^3(x^2+y^2)^{-3/2}\,dy\,dx$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{2x^2+y^2}{\sqrt{x^2+y^2}}\right]_{1/2}^{\sqrt{1-x^2}}dx =$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\frac{x^2+1}{\sqrt{1}}-\frac{2x^2+1/4}{\sqrt{x^2+1/4}}\,dx = \left[\frac{1}{3}x^3+x-\frac{1}{2}x\sqrt{4x^2+1}\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\sqrt{4}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\sqrt{4} = \boxed{\frac{\sqrt{3}}{4}}$$