Name: ______________________

Clear your desk of everything excepts pens, pencils and erasers. If you have a question raise your hand and I will come to you.

1. (2 points) **Fill-in-the-Blank. No work needed. No partial credit available.**
   Let $f(x) = \frac{x}{x^2+1}$.
   - The interval(s) on which $f(x)$ is increasing are $(-1, 1)$.
   - The interval(s) on which $f(x)$ is concave up are $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

   Hint: For the second part, once you've differentiated, don't start by trying to multiply out the numerator – factor out a copy of (x^2+1) first.

Extra Work Space.

Increasing - $f'(x) > 0$ on the interval.

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

$\leftarrow$ 0 at $x = 1, -1$

$\leftarrow$ Always positive

We see $f'(x)$ can change sign at $x = 1, -1$, so we have intervals as follows:

Interval	Sign of $f'(x)$
$(-\infty, -1)$	$-$
$(-1, 1)$	$+$
$(1, \infty)$	$-$

Concave up - $f''(x) > 0$ on the interval

0 at $x=0$, $x = \pm\sqrt{3}$

$$f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)4x(x^2+1)}{(x^2+1)^4} = \frac{-2x(x^2+1)[x^2+1+2(1-x^2)]}{(x^2+1)^4} = \frac{-2x(x^2+1)[3-x^2]}{(x^2+1)^4}$$

$\nwarrow$ Always positive

We see $f''(x)$ can change sign at $x = 0$, $x = \pm\sqrt{3}$

Interval	Sign of $f''(x)$
$(-\infty, -\sqrt{3})$	$-$
$(-\sqrt{3}, 0)$	$+$

Interval	Sign of $f''(x)$
$(0, \sqrt{3})$	$-$
$(\sqrt{3}, \infty)$	$+$

Continue on to back side

2. Let $f(x) = \begin{cases} x^2 - 1 & x < 0 \\ \frac{1}{x-1} & 0 \leq x \end{cases}$

(a) (1 point) The hypotheses of the Mean Value Theorem are true for $f(x)$ on exactly one of the intervals $[-1, 0]$ and $[0, 2]$. Which one is it? Explain your answer.

(b) (2 points) At what points c in the interval you picked in part (a) is the conclusion of the Mean Value Theorem satisfied?

(a) $F(x)$ must be continuous on $[a,b]$ and differentiable on (a,b)

• $F(x)$ is not continuous on $[0,2]$ $\rightarrow$ not defined at $x=1$.

• $F(x)$ is continuous on $[-1,0]$. The only tricky point is $x=0$, but $\lim_{x \to 0^-} x^2-1 = -1 = \frac{1}{0-1} = F(0)$. Furthermore on $(-1,0)$, $F(x) = x^2-1$ and is differentiable.

(b) Want $c \in (-1,0)$ such that $F'(c) = \frac{F(0) - F(-1)}{0+1} = \frac{-1-0}{1} = -1$

$F'(x) = 2x$ on $(-1,0)$

$2x = -1$

$x = \frac{-1}{2}$

So $c = \frac{-1}{2}$ satisfies the conclusion of the Mean Value Theorem.

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