2010-06-14

Calculus II : Midterm

Name: ______________________________

- **Write your name on this paper.**
- You have 90 minutes to complete the exam.
- The exam is worth 30 points total.
- **Please neatly write up all answers with explanations on these sheets. Just giving an answer without explanation gets zero points.**
- **Do not turn in scratch paper.** It will be ignored.
- Calculators and textbooks are **NOT** allowed during the exam.

1. Evaluate the following integrals.

(a) (5 points)

$$\int \frac{\ln t}{\sqrt{t}}\, dt$$

Let $dv = \frac{1}{\sqrt{t}}dt$ and $u = \ln t$. Then $v = 2\sqrt{t}$ and $du = \frac{1}{t}dt$. So

$$\begin{aligned}\int \frac{\ln t}{\sqrt{t}}\, dt &= \ln t \cdot 2\sqrt{t} - \int 2\sqrt{t} \cdot \frac{1}{t}\, dt \\ &= 2\sqrt{t} \cdot \ln t - 4\sqrt{t} + C\end{aligned}$$

□

(b) (5 points)

$$\int \tan^3 x\ \sec^3 x\ dx$$

Since $(\sec x)' = \tan x \sec x$ and $\tan^2 x + 1 = \sec^2 x$, let $u = \sec x$, then

$$\begin{aligned}\int \tan^3 x\ \sec^3 x\ dx &= \int \tan^2 x \cdot \sec^2 x \cdot (\tan x \sec x)\, dx \\ &= \int \left(-1 + u^2\right) u^2 du \\ &= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= -\frac{1}{3}\sec^3 x + \frac{1}{5}\sec^5 x + C\end{aligned}$$

□

(c) (10 points)

$$\int \frac{1}{\sqrt{5 - u^2 + 4u}}\, du$$

Since

$$5 - u^2 + 4u = 9 - (u-2)^2$$

let $u - 2 = 3\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $du = 3\cos\theta\ d\theta$ and

$$\sqrt{9 - (u-2)^2} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

because $\cos\theta \ge 0$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So

$$\begin{aligned}\int \frac{1}{\sqrt{5 - u^2 + 4u}}\, du &= \int \frac{\cos\theta}{\cos\theta}\, d\theta \\ &= \theta + C \\ &= \sin^{-1}\left(\frac{u-2}{3}\right) + C\end{aligned}$$

□

2. Compute the following integral. (15 points)

$$\int \frac{2x^2 - x + 1}{(x-1)(x^2+1)}\,dx$$

Since

$$\frac{2x^2 - x + 1}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{x}{x^2+1}$$

we have

$$\begin{aligned}\int \frac{2x^2 - x + 1}{(x-1)(x^2+1)}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx \\ &= \ln|x-1| + \int \frac{x}{x^2+1}\,dx\end{aligned}$$

Let $u = x^2 + 1$. Then $du = 2x\ dx$ so

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C.$$

Therefore,

$$\int \frac{2x^2 - x + 1}{(x-1)(x^2+1)}\,dx = \ln|x-1| + \frac{1}{2}\ln\left(x^2+1\right) + C$$

$\square$

3.

(a) Show that

$$\int \sqrt{1+x^2}\,dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + C$$

(10 points) Let $x = \tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and

$$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = |\sec\theta| = \sec\theta$$

since $\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So

$$\int \sqrt{1+x^2}\,dx = \int \sec^3\theta\,d\theta = \int \sec\theta\cdot\sec^2\theta\,d\theta$$

Let $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$. Then $du = \tan\theta\sec\theta\,d\theta$ and $v = \tan\theta$. So

$$\begin{aligned}\int \sec\theta\cdot\sec^2\theta\,d\theta &= \sec\theta\cdot\tan\theta - \int \tan^2\theta\sec\theta\,d\theta\\ &= \sec\theta\cdot\tan\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\,d\theta\\ &= \sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta| - \int \sec^3\theta\,d\theta.\end{aligned}$$

Therefore

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\cdot\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.$$

Since $\sec\theta = \sqrt{1+x^2}$ and $\tan\theta = x$, we have

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| + C.$$

□

(b) Find the length of the arc of the parabola $y = \frac{1}{2}x^2$ from $(0,0)$ to $(2,2)$ (10 points). Since $\frac{dy}{dx} = x$, we have

$$\begin{aligned}\text{arc length} &= \int_0^2 \sqrt{1+x^2}\,dx\\ &= \left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]_0^2\\ &= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\end{aligned}$$

□

4. Find the volume of the solid obtained by rotating the region bounded by $y = \ln x$, $y = 0$ and $x = e$ about the line $x = 3$. (15 points)

$$\begin{aligned}
\text{Volume} &= \int_0^1 \left(\pi\,(3 - e^y)^2 - \pi(3-e)^2\right)\, dy \\
&= \pi \int_0^1 \left(9 - 6e^y + e^{2y} - 9 + 6e - e^2\right)\, dy \\
&= \pi \left[-6e^y + \frac{1}{2}e^{2y} + 6ey - e^2 y\right]_0^1 = \pi\left(\frac{e^2}{2} - e^2 + 6 - \frac{1}{2}\right) \\
&= \frac{\pi}{2}\left(11 - e^2\right)
\end{aligned}$$

□

5.

(a) Find the area of the region bounded by the given curves.

$$y = \left|\cos\left(\frac{\pi x}{2}\right)\right|, \quad y = -x+1$$

(10 points)

$$\begin{aligned} \text{area} &= \int_0^1 \left(\cos\frac{\pi x}{2} - (-x+1)\right)\, dx \\ &= \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{1}{2}x^2 - x\right]_0^1 \\ &= \frac{2}{\pi} + \frac{1}{2} - 1 = \frac{2}{\pi} - \frac{1}{2} = \frac{4-\pi}{2\pi} \end{aligned}$$

□

(b) Find the volume of the solid obtained by rotating the region in (a) about y-axis (15 points).

$$\begin{aligned} \text{Volume} &= \int_0^1 2\pi x\left(\cos\left(\frac{\pi x}{2}\right) + x - 1\right)\, dx \\ &= 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)\, dx + 2\pi\int_0^1 \left(x^2 - x\right)\, dx \\ &= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^1 + 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)\, dx. \end{aligned}$$

Let $u = x$ and $dv = \cos\left(\frac{\pi x}{2}\right)\, dx$. Then $du = dx$ and $v = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)$. So

$$\begin{aligned} 2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)\, dx &= \left[2\pi x \cdot \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - \int_0^1 2\pi\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\, dx \\ &= 4 + \left[4\cdot\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 \\ &= 4 - \frac{8}{\pi} \end{aligned}$$

Therefore,

$$\text{Volume} = 4 - \frac{\pi}{3} - \frac{8}{\pi}$$

□

6. Evaluate

$$\int_{\frac{\pi}{4}}^{\pi} \tan x \; dx$$

if possible (10 points).

Since $\tan x$ is not defined at $\frac{\pi}{2}$, if

$$\lim_{t \to \frac{\pi}{2}^-} \int_{\frac{\pi}{4}}^{t} \tan x \; dx$$

and

$$\lim_{t \to \frac{\pi}{2}^+} \int_{t}^{\pi} \tan x \; dx$$

are convergent, then

$$\int_{\frac{\pi}{4}}^{\pi} \tan x \; dx$$

is also convergent. But

$$\lim_{t \to \frac{\pi}{2}^-} \int_{\frac{\pi}{4}}^{t} \tan x \; dx = \lim_{t \to \frac{\pi}{2}^-} \left(\ln |\sec t| + \ln \sqrt{2} \right)$$

is divergent. So the integral is divergent. □

7. Determine whether the following integral is convergent or divergent.

$$\int_2^\infty \frac{1}{\sqrt{x^4+1}}\,dx$$

Explain why. If you are using a theorem, quote the theorem precisely and make sure that the hypotheses of the theorem are satisfied before applying it (15 points).

Since

$$\sqrt{x^4+1} > \sqrt{x^4} = x^2$$

we have

$$0 < \frac{1}{\sqrt{x^4+1}} < \frac{1}{x^2}.$$

Here

$$\lim_{t\to\infty}\int_2^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{t}+\frac{1}{2}\right] = \frac{1}{2}.$$

By Comparison theorem (5 points for stating theorem),

$$\int_2^\infty \frac{1}{\sqrt{x^4+1}}\,dx \le \frac{1}{2}$$

is convergent. □

Trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Integration formulas Constants of integration have been omitted

$$\int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1}, & \text{if } n \neq -1 \\ \ln|x|, & \text{if } n = -1 \end{cases}$$

$$\int e^x \, dx = e^x$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \sec x \, dx = \ln|\sec x + \tan x|$$

$$\int \tan x \, dx = \ln|\sec x|$$