2010-06-14

Calculus II : Practice final (Calculators and textbooks are **NOT** allowed during the exam.)

1. Evaluate the integral

$$\int \frac{3w-1}{w+2}\, dw.$$

$$\begin{aligned}\int \frac{3w-1}{w+2}\, dw &= \int \left(3 - \frac{7}{w+2}\right)\, dw \\ &= 3w - 7\ln|w+2| + C\end{aligned}$$

□

2. Find the value of p for which the series is convergent

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}.$$

If $p \leq 0$, then

$$\lim_{n\to\infty} \frac{\ln p}{n^p} = \infty$$

so by the divergent test, this series is divergent for $p \leq 0$.

If $p = 1$, then

$$\frac{\ln n}{n} \geq \frac{1}{n}$$

for any positive integer n. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, by the comparison test, this series is also divergent.

For $p > 0$ and $p \neq 1$, let

$$f(x) = \frac{\ln x}{x^p}$$

for $x \geq 1$. Then

- $f(x)$ is positive for $x > 1$ and continuous on $[1, \infty)$
- since

$$f'(x) = \frac{1}{x^{p+1}} - p\frac{\ln x}{x^{p+1}} = \frac{1}{x^{p+1}}(1 - p\ln x)$$

  so $f'(x) < 0$ for $x > e^{1/p}$. So $f(x)$ is decreasing for $(e^{1/p}, \infty)$.
- for each integer $n \geq 1$,

$$f(n) = \frac{\ln n}{n^p}$$

By the integral test, this series is convergent if and only if $\int_1^{\infty} f(x)\, dx$ is convergent. Use integration by part, let $u = \ln x$ and $dv = \frac{1}{x^p}\, dx$ (then $v = \frac{1}{1-p}\frac{1}{x^{p-1}}$ if $p \neq 1$)

$$\begin{aligned}\int_1^t \frac{\ln x}{x^p}\, dx &= \left[\frac{1}{1-p}\frac{\ln x}{x^{p-1}}\right]_1^t - \int_1^t \frac{1}{1-p}\frac{1}{x^p}\, dx \\ &= \frac{1}{1-p}\frac{\ln t}{t^{p-1}} - \frac{1}{(1-p)^2}\left(\frac{1}{t^{p-1}} - 1\right)\end{aligned}$$

Then the limit $\int_1^\infty \frac{\ln x}{x^p}\ dx$ is convergent whenever $p - 1 > 0$.

Therefore, this series is convergent if and only if $p > 1$. $\square$

3. Find the surface area of the surface obtained by rotating

$$y = \sqrt{7 - x^2}, \qquad 0 \le x \le 1$$

about the x-axis.

Surface area S:

- about x-axis:
$$S = \int 2\pi y\ ds$$
- about y-axis:
$$S = \int 2\pi x\ ds$$

where

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

This case,

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{7 - x^2}}$$

so

$$\begin{aligned} S &= 2\pi \int_0^1 \sqrt{7 - x^2} \cdot \sqrt{1 + \left(\frac{-x}{\sqrt{7 - x^2}}\right)^2}\ dx \\ &= 2\pi \int_0^1 \sqrt{7}\ dx = 2\sqrt{7}\pi \end{aligned}$$

$\square$

4.

(a) Sketch the curve

$$r = \cos 2\theta.$$

Four-leaved rose. (p. 643)

(b) Find the points on the curve $r = \cos 2\theta$ where the tangent line is horizontal or vertical.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = -2\sin 2\theta$$

- $\frac{dy}{d\theta} = 0$

$$\begin{aligned} &\Rightarrow \frac{dy}{d\theta} = -2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = 0 \\ &\Rightarrow -4\sin^2\theta\cos\theta + (-1 + 2\cos^2\theta)\cos\theta = 0 \end{aligned}$$

$$\Rightarrow \cos\theta(6\cos^2\theta - 5) = 0$$
$$\Rightarrow \cos\theta = 0 \text{ or } \cos\theta = \pm\frac{\sqrt{5}}{\sqrt{6}}$$

- $\frac{dx}{d\theta} = 0$

$$\Rightarrow \frac{dx}{d\theta} = -2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = 0$$
$$\Rightarrow \sin\theta(6\sin^2\theta - 5) = 0$$
$$\Rightarrow \sin\theta = 0 \text{ or } \sin\theta = \pm\frac{\sqrt{5}}{\sqrt{6}}$$

Forget this problem... it is complicated to compute. But you should know how to get the horizontal and vertical tangents... (use $\frac{dx}{d\theta} = 0$ or $\frac{dy}{d\theta} = 0$) □

(c) Find all points of intersection of the curve $r = \cos 2\theta$ and $r = \frac{1}{2}$.

$$\cos 2\theta = \frac{1}{2} \text{ or } \cos 2\theta = -\frac{1}{2}$$
$$\Rightarrow 2\theta = \pm\frac{\pi}{3} + 2n\pi \text{ or } \pm\frac{2\pi}{3} + 2n\pi$$
$$\Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$$
$$\Rightarrow \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, -\frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$$
$$\left(-\frac{1}{2}, \frac{\pi}{3}\right), \left(-\frac{1}{2}, \frac{4\pi}{3}\right), \left(-\frac{1}{2}, -\frac{\pi}{3}\right), \left(-\frac{1}{2}, \frac{2\pi}{3}\right)$$

□

(d) Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2 2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 4\theta)\, d\theta$$
$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{8}$$

□

5. Let $x = e^t + e^{-t}$ and $y = 1 - 2t$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
Since $\frac{dy}{dt} = -2$ and $\frac{dx}{dt} = e^t - e^{-t}$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2}{e^t - e^{-t}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{2(e^t+e^{-t})}{(e^t-e^{-t})}}{(e^t - e^{-t})^2} = \frac{2(e^t + e^{-t})}{(e^t - e^{-t})^3}$$

(b) For $0 \le t \le 3$, find the exact length of the curve.

$$\begin{aligned} \text{length} &= \int_0^3 \sqrt{(e^t - e^{-t})^2 + 4}\, dt = \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt \\ &= \int_0^3 \left(e^t + e^{-t}\right)\, dt = \left[e^t - e^{-t}\right]_0^3 \\ &= e^3 - \frac{1}{e^3} \end{aligned}$$

□

6. Determine whether the series is conditionally convergent, absolutely convergent, or divergent. Explain why.

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

By the alternating series test, it is convergent. But by p-series, it is not absolutely convergent. So it is conditionally convergent.

(b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^{\frac{1}{n}}}$$

Since

$$\lim_{n\to\infty} \frac{1}{2^{\frac{1}{n}}} = 1$$

by the divergent test, this series is divergent.

(c)

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

By the Root test,

$$\lim_{n\to\infty} \left(\left| \frac{(n!)^n}{n^{4n}} \right| \right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n!}{n^4} \right) = \lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) (n-4)! \right]$$

is divergent. So the series is divergent.

7. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

By the theorem for the power series, this series

- converges for at least $-4 \le x < 4$
- diverges for at least $x < -6$ and $x \ge 6$

(a)

$$\sum_{n=0}^{\infty} c_n$$

$x = 1$, so converges.

(b)

$$\sum_{n=0}^{\infty} c_n 8^n$$

$x = 8 > 6$, so diverges.

(c)

$$\sum_{n=0}^{\infty} c_n (-3)^n$$

$x = -3$, so converges.

8. Find the radius of convergence and interval of convergence of the series

$$f(x) = \sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{\sqrt{n}}.$$

By the Ration test,

$$\left| \frac{\frac{2^{n+1}(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{2^n (x-1)^n}{\sqrt{n}}} \right| = 2\,|x-1| \frac{\sqrt{n}}{\sqrt{n+1}} \to 2\,|x-1|$$

as $n \to \infty$. The series is convergent if $2\,|x-1| < 1$. So the radius of convergence is $\frac{1}{2}$.

At points $x - 1 = \pm\frac{1}{2}$,

- $x = \frac{3}{2}$:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is divergent.

- $x = -\frac{1}{2}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

is convergent, by the alternating series theorem.

So the interval of convergence is

$$\left[-\frac{1}{2}, \frac{3}{2}\right).$$

□

9.

(a) Find the radius of convergence and interval of convergence of the seires

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By the Ratio test,

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \frac{|x|}{n+1} \to 0$$

as $n \to \infty$ for any x. So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$. □

(b) In fact,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for x in the interval of convergence obtained in (a). Evaluate the integral

$$\int e^{-x^2}\, dx.$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

So

$$\int e^{-x^2}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1} + C$$

10. Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}.$$

What is the radius of convergence?

$$\left(\frac{-1}{1+x}\right)' = \frac{1}{(1+x)^2}$$

$$\frac{-1}{1+x} = \sum_{n=0}^{\infty} (-1)(-x)^n$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} (-1)^{n+1} x^n\right)' = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

By the Ratio text

$$\left|\frac{x^{n+1}(n+2)}{x^n(n+1)}\right| \to |x|$$

as $n \to \infty$. So the radius of convergence is 1.

Trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Integration formulas Constants of integration have been omitted

$$\int \frac{1}{x^n}\,dx = \begin{cases} \frac{1}{(-n+1)x^{n-1}}, & \text{if } n \neq 1 \\ \ln|x|, & \text{if } n = 1 \end{cases}$$

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \sec x \tan x\,dx = \sec x$$

$$\int \sec x\,dx = \ln|\sec x + \tan x|$$

$$\int \tan x\,dx = \ln|\sec x|$$