S1102.D Summer 2011
Calculus II

Midterm

Instructions: You have 90 minutes to complete the exam. There are seven problems, worth a total of 120 points. Calculators and textbooks are not allowed. Provide the answers in the simplest possible form that does not require calculator use. (E.g. expressions like $\sqrt{13}$ are fine.) Show all of your work: if you only give the answer you will receive no credit, but conversely, partial credit will be given for partial solutions.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	20	
2	10	
3	25	
4	15	
5	15	
6	15	
7	0	
Total:	100	

Problem 1.

Evaluate the following integrals.

(a) [5pts.]

$$\int x^2 2^x dx$$

Solution: We use rapid integration by parts. Let our first u be x^2 and first dv be $2^x dx$.

u	dv
x^2	2^x
$2x$	$\frac{2^x}{\ln(2)}$
2	$\frac{2^x}{\ln(2)^2}$
0	$\frac{2^x}{\ln(2)^3}$

Ergo

$$\begin{aligned}\int x^2 2^x dx &= x^2 \frac{2^x}{\ln(2)} - 2x\frac{2^x}{\ln(2)^2} + 2\frac{2^x}{\ln(2)^3} + C \\ &= \frac{2^x(x^2\ln(2)^2 - 2x\ln(2) + 2)}{\ln(2)^3} + C\end{aligned}$$

(b) [5pts.]

$$\int \sin^3(\theta)\cos^2(\theta) d\theta$$

Solution: We use the Pythagorean identity $\sin^2(\theta) = 1 - \cos^2(\theta)$ to compute

$$\begin{aligned}\int \sin^3(\theta)\cos^2(\theta) d\theta &= \int \sin\theta(\sin^2\theta)\cos^2\theta d\theta \\ &= \int \sin\theta(1-\cos^2\theta)\cos^2\theta d\theta\end{aligned}$$

Make a substitution $u = \cos\theta$, so that $du = -\sin\theta d\theta$ to obtain

$$\begin{aligned}\int \sin\theta(1-\cos^2\theta)\cos^2\theta d\theta &= \int (1-u^2)u^2(-du)\\ &= \int (u^4-u^2)du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C\end{aligned}$$

(c) [10pts.]

$$\int \frac{1}{(u^2+6u+25)^{\frac{3}{2}}}du$$

Solution: We complete the square in the denominator, observing that $u^2 + 6u + 25 = (u^2 + 6u + 9) + 16 = (u+3)^2 + 16$. Let $t = u+3$, so that $dt = du$ and the integral becomes

$$\begin{aligned}\int \frac{1}{(u^2+6u+25)^{\frac{3}{2}}}du &= \int \frac{1}{((u+3)^2+16)^{\frac{3}{2}}}du\\ &= \int \frac{1}{(t^2+16)^{\frac{3}{2}}}\end{aligned}$$

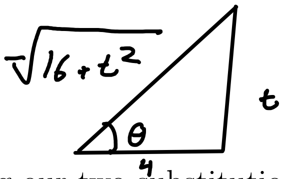
Now we are well-placed to use trigonometric substitution. Let $t = 4\tan\theta$, with $\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $dt = 4\sec^2\theta$. Then our integral becomes

$$\begin{aligned}\int \frac{1}{(t^2+16)^{\frac{3}{2}}} &= \int \frac{4\sec^2\theta d\theta}{(16\tan^2\theta+16)^{\frac{3}{2}}}\\ &= \int \frac{4\sec^2\theta d\theta}{(16\sec^2\theta)^{\frac{3}{2}}}\\ &= \int \frac{4\sec^2\theta d\theta}{(4|\sec\theta|)^3}\end{aligned}$$

Because $\sec\theta$ is positive on $\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $|\sec\theta| = \sec\theta$, and so we can finish as below.

$$\begin{aligned}\int \frac{4\sec^2\theta d\theta}{(4|\sec\theta|)^3} &= \int \frac{\sec^2\theta d\theta}{16\sec^3\theta} \\ &= \frac{1}{16}\int \cos\theta d\theta \\ &= \frac{1}{16}\sin\theta + C\end{aligned}$$

Since $\tan\theta = \frac{t}{4}$, we see from the diagram below that $\sin\theta = \frac{t}{\sqrt{16+t^2}}$.



We finish by reversing our two substitutions.

$$\begin{aligned}\int \frac{1}{(u^2+6u+25)^{\frac{3}{2}}}du &= \frac{1}{16}\sin\theta + C \\ &= \frac{t}{16\sqrt{t^2+16}} + C \\ &= \frac{u+3}{16\sqrt{(u+3)^2+16}} + C \\ & \frac{u+3}{16\sqrt{u^2+6x+25}} + C\end{aligned}$$

Problem 2. *10pts.*

Evaluate the following integral.

$$\int \frac{3x+3}{x^3-1}dx$$

Note: $x^3 - 1 = (x-1)(x^2+x+1)$.

Solution: We use integration by partial fractions. Since $x^3-1 = (x-1)(x^2+x+1)$, we know we can write the integrand as

$$\frac{3x+3}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

for some constants A, B, and C. Therefore

$$\frac{3x+3}{x^3-1} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{x^2+x+1}$$

Set the numerators equal to get $3x+3 = A(x^2+x+1)+(Bx+C)(x-1)$. With $x=1$ we see that $6x = A(3)+0$, so $A=2$. Then when $x=0$, $3 = 2(1)+(C)(-1)$, so $1=-C$, hence $C=-1$. Finally, multiplying out we have

$$3x+3 = (A+B)x^2+(A-B+C)x+(A-C)$$

So since equality of polynomials is equality of coefficients in each degree, $A+B=0$, hence $B=-2$. All of which shows that

$$\begin{aligned}\int \frac{3x+3}{x^3-1}dx &= \int \frac{2dx}{x-1} + \int \frac{-2x-1}{x^2+x+1}dx \\ &= 2\ln|x-1| + \int \frac{-2x-1}{x^2+x+1}dx\end{aligned}$$

Let $u = x^2+x+1$, and $du = 2x+1$. Then the remaining integral becomes

$$\begin{aligned}\int \frac{3x+3}{x^3-1}dx &= 2\ln|x-1| + \int \frac{-2x-1}{x^2+x+1}dx \\ &= 2\ln|x-1| + \int \frac{-du}{u} \\ &= 2\ln|x-1| + -\ln|u| + C \\ &= 2\ln|x-1| - \ln|x^2+x+1| + C \\ &= \ln\left|\frac{(x-1)^2}{x^2+x+1}\right| + C\end{aligned}$$

Problem 3.

Suppose the height of a flying bird above the ground at position x is given by $y = h(x) = \sqrt{4-x^2}$. (So that the x-axis lies along the ground, and the bird's height is given by y.)

(a) [12pts.] Find the average height of the bird as it flies from $(0,2)$ to $(2,0)$.

Solution: The average height is $\frac{1}{2-0}\int_0^2 \sqrt{4-x^2}dx$. Let $x = 2\sin\theta$, with $dx = 2\cos\theta d\theta$, such that the integral becomes

$$
\begin{aligned}
\frac{1}{2-0}\int_0^2 \sqrt{4-x^2}dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}2\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{4\cos^2\theta}\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\cos\theta\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta \\
&= \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\
&= \left(\frac{\pi}{2} + \frac{1}{2}(0)\right) - (0+0) \\
&= \frac{\pi}{2}
\end{aligned}
$$

(b) [13pts.] Find the distance travelled by the bird along the same interval.

Solution: We see $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$, so

$$
\begin{aligned}
ds &= \sqrt{1+(\frac{dy}{dx})^2)}dx \\
&= \sqrt{1+\frac{x^2}{4-x^2}}dx \\
&= \sqrt{\frac{4-x^2+x^2}{4-x^2}}dx \\
&= \sqrt{\frac{4}{4-x^2}}dx \\
&= \frac{2}{\sqrt{4-x^2}}dx
\end{aligned}
$$

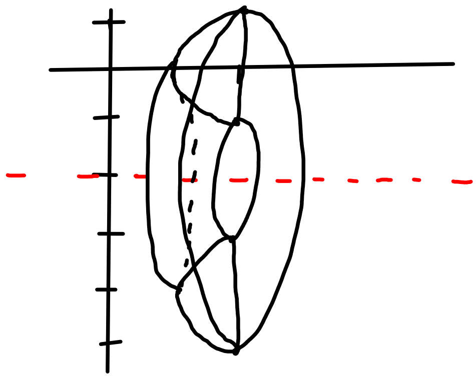
So the length of the curve traced out by the bird is

$$
\begin{aligned}
\int_0^2 ds &= \int_0^2 \frac{2}{\sqrt{4-x^2}}dx \\
&= \int_0^2 \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}dx \\
&= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 \\
&= \left(2\sin^{-1}(1) - 2\sin^{-1}(0)\right) \\
&= 2\left(\frac{\pi}{2} - 0\right) \\
&= \pi
\end{aligned}
$$

Problem 4. *15pts.*

Find the volume of the solid generated by rotating the region bounded by $x = y^2 + 1$ and $x = 2$ about the axis $y = -2$.

Solution: Our solid has the following form.



We evaluate using cylindrical shells. An approximating rectangle parallel to $y = -2$ has height $h = 2 - (y^2 + 1)$ and radius $r = y + 2$. Therefore the volume of the solid is

$$
\begin{aligned}
V = \int_{-1}^{1} 2\pi r h dy &= \int_{-1}^{1} 2\pi(y+2)(1-y^2)dy \\
&= \int_{-1}^{1} 2\pi(2+y-2y^2-y^3)dy \\
&= 2\pi \int_{-1}^{1}(2-2y^2)dy + \int_{-1}^{1}(y-y^3)dy \\
&= 2\pi\left[2\int_{0}^{1}(2-2y^2)dy + 0\right] \\
&= 4\pi\left[2y - \frac{2}{3}y^3\right]_0^1 \\
&= 4\pi\left[2-\frac{2}{3}\right] \\
&= 4\pi\left[\frac{4}{3}\right] \\
&= \frac{16\pi}{3}
\end{aligned}
$$

Problem 5.

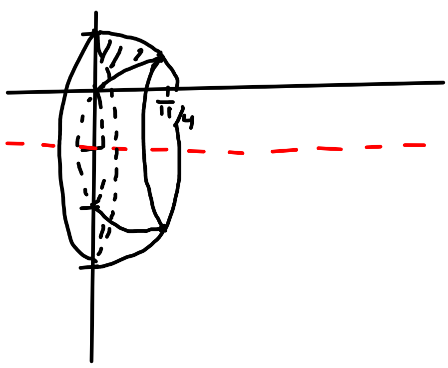
(a) [5pts.] Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{4}$.

Solution: On the interval specified cosine is greater than sine, so the area is

$$
\begin{aligned}
A &= \int_{0}^{\frac{\pi}{4}} [\cos\theta - \sin\theta]d\theta \\
&= [\sin\theta + \cos\theta]_0^{\frac{\pi}{4}} \\
&= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - [0+1] \\
&= \sqrt{2} - 1
\end{aligned}
$$

(b) [10pts.] Find the volume of the solid generated by rotating the region in (a) about the line $y = -1$.

Solution:
Solid is drawn below.



We use the disk-washer method with an outer radius of $\cos\theta + 1$ and an inner radius of $\sin\theta + 1$. The volume of the solid is

$$\begin{aligned}
V &= \int_0^{\frac{\pi}{4}} \pi[(\cos\theta+1)^2 - (\sin\theta+1)^2]d\theta \\
&= \pi \int_0^{\frac{\pi}{4}} \left[\left(\cos^2\theta + 2\cos\theta + 1\right) - \left(\sin^2\theta + 2\sin\theta + 1\right)\right] d\theta \\
&= \pi \int_0^{\frac{\pi}{4}} \left[\left(\cos^\theta - \sin^2\theta\right) + 2\cos\theta - 2\sin\theta\right] d\theta \\
&= \pi \int_0^{\frac{\pi}{4}} [\cos(2\theta) + 2\cos\theta - 2\sin\theta]\, d\theta \\
&= \pi \left[\frac{1}{2}\sin(2\theta) + 2\sin\theta + 2\cos\theta\right]_0^{\frac{\pi}{4}} \\
&= \pi \left[\left(\frac{1}{2} + \sqrt{2} + \sqrt{2}\right) - (0+0+2)\right] \\
&= \pi\left(2\sqrt{2} - \frac{3}{2}\right)
\end{aligned}$$

Problem 6. *15pts.*

Evaluate the following integral, if possible.

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

Solution:

We have

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

Let $u = \sqrt{x}$, with $du = \frac{dx}{2\sqrt{x}}$. Then

$$
\begin{aligned}
\int_0^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= \int_1^{\sqrt{t}} e^{-u} 2du \\
&= -2e^{-u}]_1^{\sqrt{t}} \\
&= -2e^{-\sqrt{t}} - (\frac{2}{e})
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= \lim_{t\to\infty} -e^{-\sqrt{t}} + \frac{2}{e} \\
&= \frac{2}{e} \\
&= \frac{2}{e}
\end{aligned}
$$

Problem 7.

Extra credit: A group of engineers is building a parabolic reflector dish whose shape will be formed by rotating the curve $y = ax^2$ between $x = 0$ m and $x = 1$ m about the y-axis. What is the surface area of the resulting dish?

Solution: We see $\frac{dy}{dx} = 2ax$, so $ds = \sqrt{1 + 4a^2x^2}$. Hence surface area is

$$
\begin{aligned}
SA &= \int 2\pi x ds \\
&= \int_0^1 2\pi x\sqrt{1 + 4a^2x^2} dx
\end{aligned}
$$

Let $u = 1 + 4a^2x^2$, so that $du = 8a^2x$. Then the integral becomes

$$
\begin{aligned}
SA &= \int_1^{1+4a^2} \frac{\pi}{4a^2}\sqrt{u} du \\
&= \left[\frac{\pi}{6a^2} u^{\frac{3}{2}}\right]_0^{1+4a^2} \\
&= \left[\frac{\pi}{6a^2}\left(\left(1 + 4a^2\right)^{\frac{3}{2}} - 1\right)\right]
\end{aligned}
$$