Arc Length & Surface Area.

Previously

Area $\xrightarrow{\text{3-d analog}}$ Volume

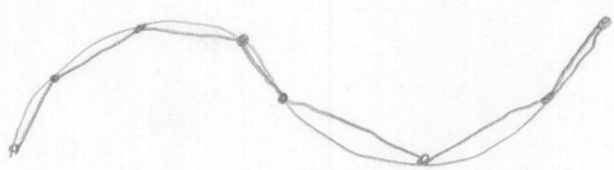
Now

Arc Length $\xrightarrow{\text{2-d analog}}$ Surface area

What do I mean by length.

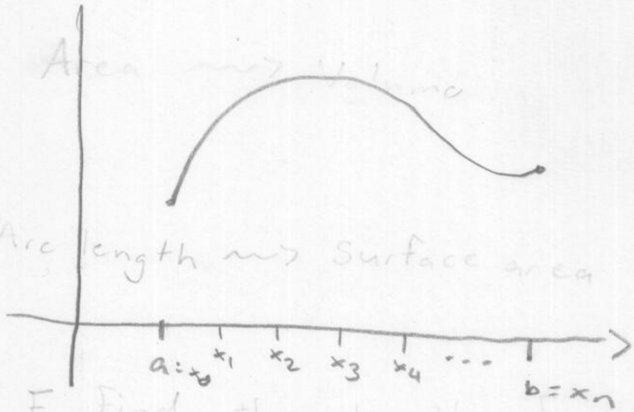
- Understand length of a line segment

- Approximate a curve by line segments.



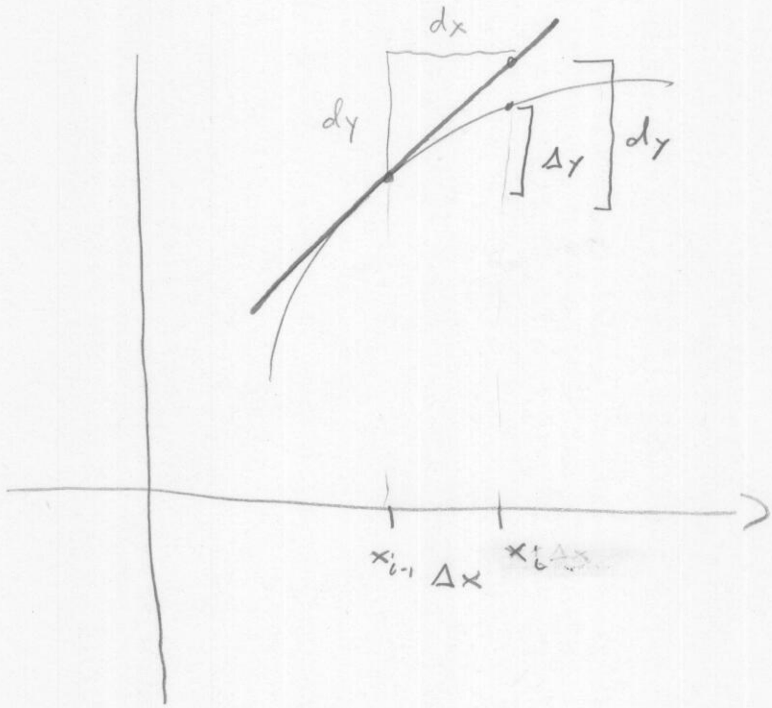
But curves come with a built in approximation, the tangent line.

Divide a curve over $[a, b]$



$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$$

Recall: The tangent line to a curve at a point approximates the curve at that point.



$$\Delta x = dx \qquad \Delta y \approx dy = f'(x_i)\,dx$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The length of the line

$$(y - y_{i-1}) = f'(x_i)(x - x_{i-1})$$

above $[x_{i-1}, x_i]$

is $\sqrt{dx^2 + dy^2}$.

$$\text{Length of line} = \sqrt{dx^2 + (dy)^2}$$

$$= \sqrt{dx^2 + (f'(x_i)\,dx)^2}$$

$$= \sqrt{dx^2(1 + f'(x_i)^2)}$$

$$= dx\sqrt{1 + [f'(x_i)]^2}$$

So the length of the curve above $[x_{i-1}, x_i]$ is approximated by $dx\sqrt{1+[f'(x_i)]^2} = \Delta x\sqrt{1+[f'(x_i)]^2}$.

Total length of curve $\approx \sum_{i=1}^{n} \Delta x\sqrt{1+[f'(x_i)]^2}$

Defn The length of a curve $f(x)$ from $x=a$ to $x=b$ is

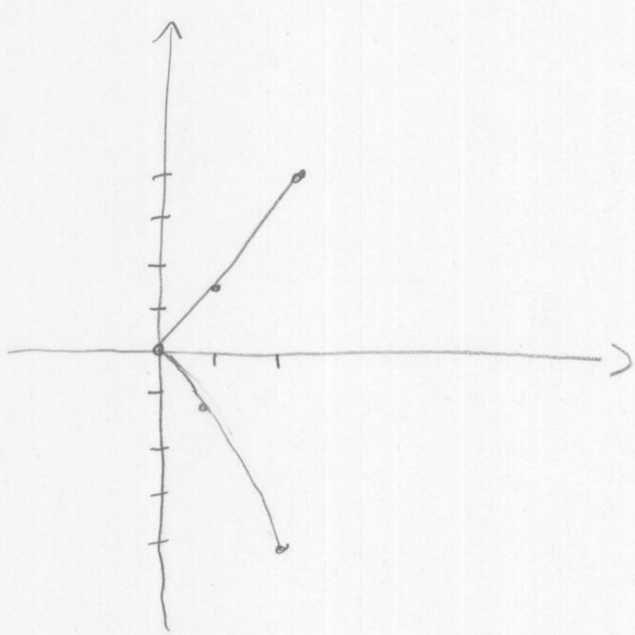
$$\lim_{n\to\infty} \sum_{i=0}^{\infty} \Delta x\sqrt{1+[f'(x_i^*)]^2} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_a^b \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{\text{"}ds\text{"}}$$

when f has a cts derivative.

Example 1

Find the length of the arc of the curve $y^2 = 2x^3$ between $(0,0)$ and $(2,4)$.



$y = \sqrt{2}x^{3/2}$ on this interval

$$\frac{dy}{dx} = \frac{3}{\sqrt{2}}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{2}x$$

$$L = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^2 \sqrt{1+\frac{9}{2}x}\,dx$$

$u = 1+\frac{9}{2}x$

$du = \frac{9}{2}dx$

$$= \int_1^{10} \frac{2}{9}\sqrt{u}\,du$$

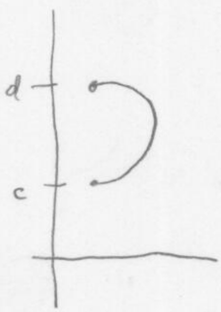
$$= \frac{2}{9}\left(\frac{2}{3}\right) u^{3/2} \Big|_1^{10}$$

$$= \frac{4}{27}\left((10)^{3/2} - 1\right)$$

$$\approx 9.54 \text{ units}$$

Of course the same computation shows if $x = g(y)$,

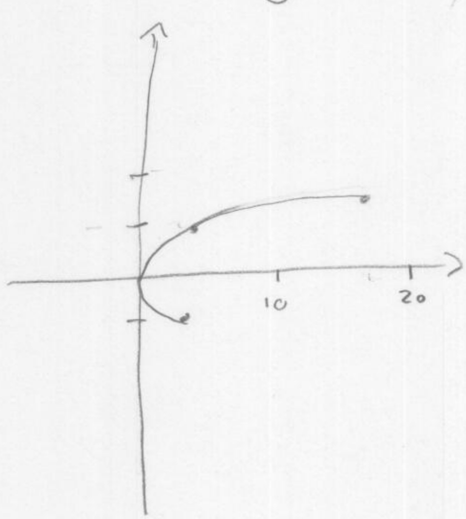
$$L = \int_c^d \overbrace{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}^{= ds} \, dy$$



$$= \int_c^d \sqrt{1 + [g'(x)]^2} \, dy$$

Example 2

Find the length of $x = 4y^2$ from $(0, 0)$ to $(16, 2)$



$$\frac{dx}{dy} = 8y$$

$$\left(\frac{dx}{dy}\right)^2 = 64y^2$$

$$L = \int_0^2 \sqrt{1 + 64y^2} \, dy$$

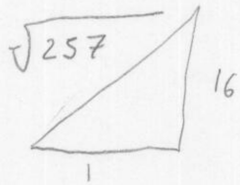
$$y = \frac{1}{8}\tan\theta \qquad dy = \frac{1}{8}\sec^2\theta \, d\theta$$

$$= \int_0^{\tan^{-1}(16)} \sqrt{1 + \tan^2\theta} \cdot \frac{1}{8}\sec^2\theta \, d\theta$$

$$= \int_0^{\tan^{-1}(16)} \frac{1}{8}\sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) \Big|_0^{\tan^{-1}(16)}$$

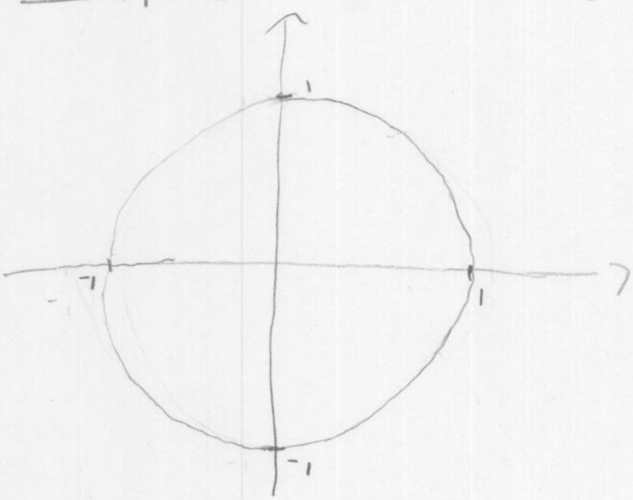
using a formula I <u>don't</u> expect you to have memorized



$$= \frac{1}{2}\left[\sqrt{257}\,(16) + \ln|\sqrt{257} + 16|\right] - \frac{1}{2}\left[1(0) + \ln|1+0|\right]$$

$$= \frac{1}{2}\left[16\sqrt{257} + \ln|\sqrt{257}+16|\right] \text{ units.}$$

<u>Example</u> Circumference of a circle



Four times the length in quadrant 1.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}(-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$

$$L = 4\int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$$

$$= 4\int_0^1 \sqrt{\frac{1}{1-x^2}}\,dx$$

$$= 4\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= 4\sin^{-1}x \Big|_0^1$$

$$= 4\left[\sin^{-1}1 - \sin^{-1}0\right]$$

$$= 4\left[\frac{\pi}{2} - 0\right]$$

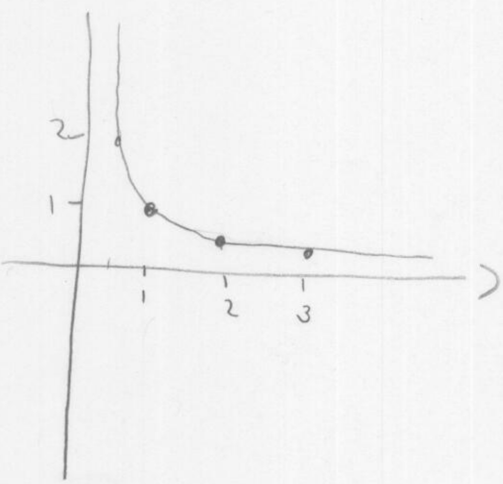
$$= 2\pi$$

Approximations

Because arclength is tricky (we've seen sums under square roots are a problem) we often approximate integrals numerically.

Length of a hyperbola

Find length of $xy = 1$ from $(1,1)$ to $(3,\frac{1}{3})$



$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-1}{x^2} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4}$$

$$L = \int_1^3 \overbrace{\sqrt{1 + \frac{1}{x^4}}}^{g}\, dx$$

4 subintervals $\qquad \Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

Midpoints, for example

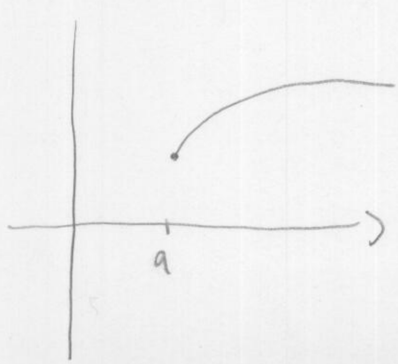
$$L \approx \frac{1}{2}\left[g\left(\tfrac{5}{4}\right) + g\left(\tfrac{7}{4}\right) + g\left(\tfrac{9}{4}\right) + g\left(\tfrac{11}{4}\right)\right]$$

$$= \frac{1}{2}\left[\sqrt{1+\left(\tfrac{4}{5}\right)^4} + \sqrt{1+\left(\tfrac{4}{7}\right)^4} + \sqrt{1+\left(\tfrac{4}{9}\right)^4} + \sqrt{1+\left(\tfrac{4}{11}\right)^4}\right]$$

$$= 2.13 \text{ units}$$

Arc Length as a Function.

We're frequently interested in questions of the form "How far has a particle travelled since such-and-such a time?

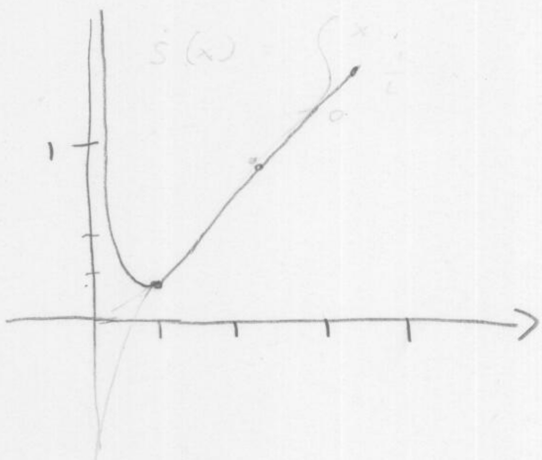


Want a function that describes length of the curve from a to x?

By convention this function is $s(x)$.

Example Find an arc length function for

$y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ st $s(1) = 0$.



$$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x}$$

$$= \frac{x^2-1}{2x}$$

$$\left[\frac{dy}{dx}\right]^2 = \frac{x^4-4x^2+1}{4x^2}$$

$$s(x) = \int_0^x \sqrt{1 + \frac{t^4-4t^2+1}{4t^2}}\,dt$$

$$= \int_0^x \sqrt{\frac{t^2+4t^2+1}{4t^2}}\,dt$$

$$= \int_0^x \sqrt{\frac{(t^2+1)^2}{(2t)^2}}\,dt$$

$$= \int_0^x \frac{(t^2+1)}{2t}\,dt$$

$$= \int_0^x \left[\frac{t}{2} + \frac{1}{2t}\right]dt$$

$$= \frac{1}{2}\left[\frac{1}{2}t^2 + \ln|t|\right]_0^x$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2 + \ln|x|\right] - \frac{1}{2}\left[0+1\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2 + \ln|x| - 1\right]$$

Distance to 2

$$s(2) = \frac{1}{2}\left[2 + \ln 2 - 1\right]$$

$$= \frac{1}{2}\left[1 + \ln 2\right]$$

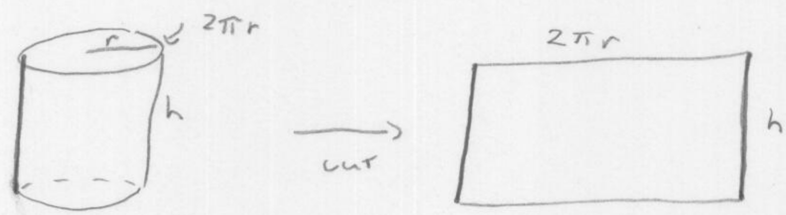
Can compute easily for any number now.

Surface area of a Surface of Revolution (w/ cross-sections circles)

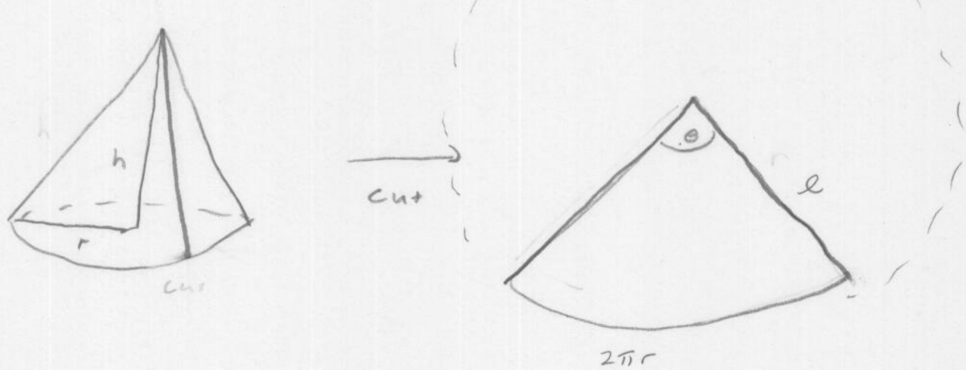
Area		Volume	Arc Length		Surface Area
Rectangles	⇝	Rotations of rectangles (= cylinders)	Line segments	⇝	Rotations of line segments (bands)

As before we build from simple cases.

Circular cylinder



$SA = 2\pi r h$

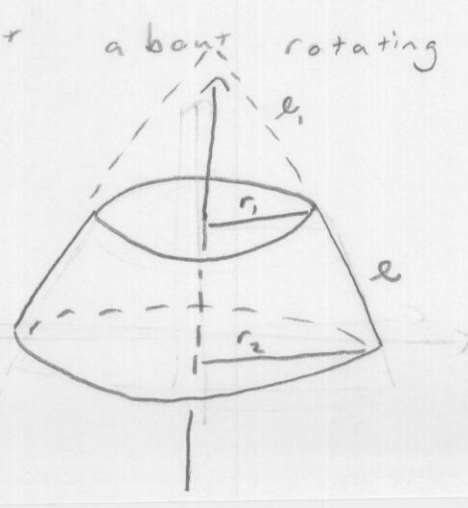


$$\theta = \frac{2\pi r}{l}$$

$$\text{Area} = \pi l^2 \cdot \left(\frac{\theta}{2\pi}\right)$$

$$= l \pi r$$

What about rotating just a line segment?



Similar triangles

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

$$r_2 l_1 = r_1(l_1 + l)$$

$SA = [\text{SA of big cone}] - [\text{SA little cone}]$

$= (\ell + \ell_1) r_2 \pi - \ell_1 r_1 \pi$

$= \pi [r_2 \ell + r_2 \ell_1 - \ell_1 r_1]$

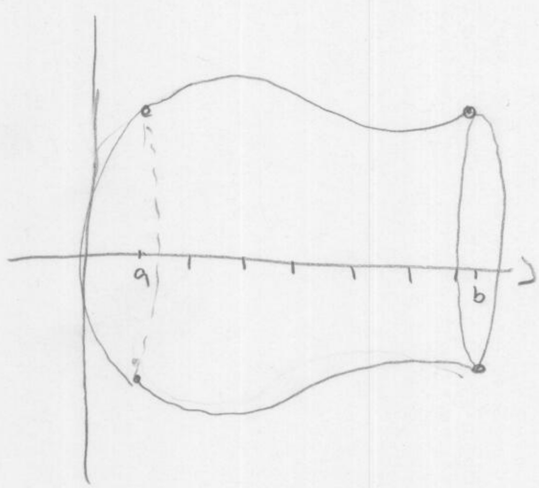
$= \pi [r_2 \ell + r_1 (\ell_1 + \ell) - \ell_1 r_1]$

$= \pi [r_2 \ell + r_1 \ell]$

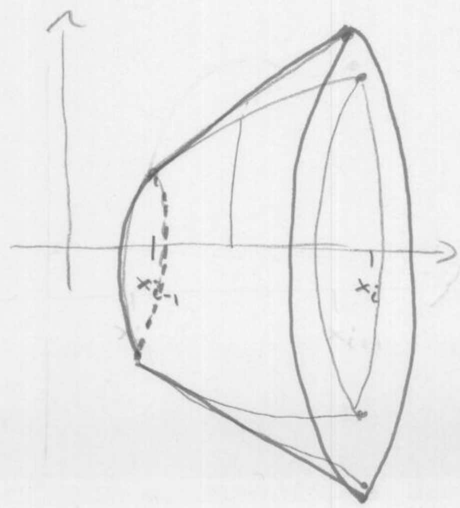
$= \pi \ell (r_1 + r_2)$

If the line segment is quite short, so that $r_1 \sim r_2$, then we sa SA of the band $= 2\pi r \ell$

So suppose I rotate a curve.



Divide into regions.



- Length of line segment approximating curve is $\Delta x \sqrt{1 + f'(x_{i-1})^2}$
- SA of band approximating SA of shape around $[x_{i-1}, x_i]$ is $\approx$ $2\pi f(x) \sqrt{1 + f'(x_{i-1})^2}\, \Delta x$

Total SA $\approx \sum_{i=1}^{n} f(x_{i-1}) \sqrt{1+[f'(x_{i-1})]^2}\, dx$

Defn The surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis is

$$SA = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}) \sqrt{1 + [f'(x_{i-1})]^2}\, dx$$

$$= \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

when f is positive & has cts derivative.

IF the curve is of the form $x = g(y)$, $c \le y \le d$, this integral is

$$SA = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

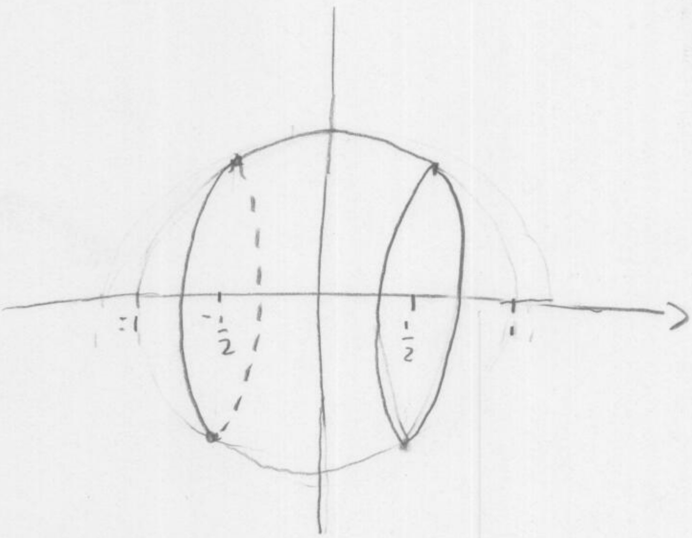
Notice this is much simpler!

In both cases we loosely refer to $SA = \int 2\pi y\, ds$.

Similarly if we rotate around the x-axis, $SA = \int 2\pi x\, ds$.

Example 1

Find the area of the surface generated by rotating the arc of a circle of radius 1 from $-\frac{1}{2} \le x \le \frac{1}{2}$ above the x-axis about the x-axis.



$$SA = \int 2\pi y \, ds$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\pi \sqrt{1-x^2} \cdot \sqrt{\frac{1}{1-x^2}} \, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\pi \, dx$$

$$= 2\pi x \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= 2\pi$$

$$y = \sqrt{1-x^2}$$

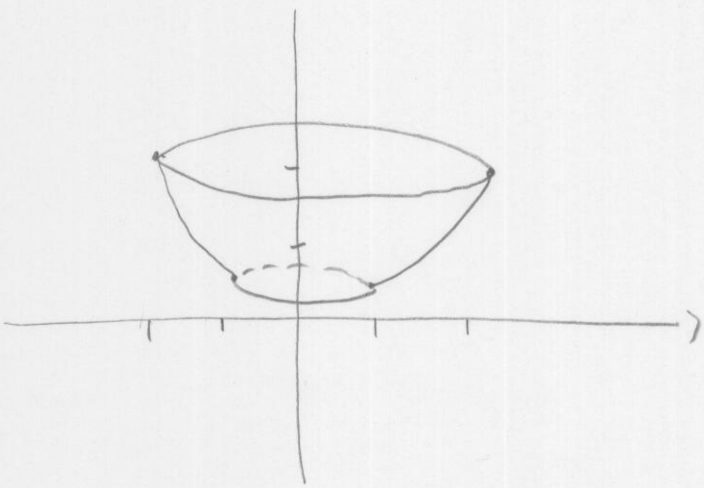
$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1 + \frac{x^2}{1-x^2}} \, dx$$

$$= \sqrt{\frac{1}{1-x^2}} \, dx$$

Example 2

Rotate the curve $y = \frac{1}{2}x^2$ from $(1, \frac{1}{2})$ to $(2, 2)$ about the y-axis. Find resulting surface area.



$$\frac{dy}{dx} = x$$

$$\left(\frac{dy}{dx}\right)^2 = x^2$$

$$ds = \sqrt{1 + x^2}\, dx$$

$$SA = \int 2\pi x \, ds$$

$$= \int_1^2 2\pi x \sqrt{1 + x^2}\, dx \qquad u = 1 + x^2, \quad du = 2x\,dx$$

$$= \int_2^5 \pi \sqrt{u}\, du$$

$$= \pi \left(\frac{2}{3}\right) u^{3/2} \Big|_2^5$$

$$= \frac{2\pi}{3}\left[5^{3/2} - 2^{3/2}\right]$$

OR

$$x = \sqrt{2y}$$

$$\frac{dx}{dy} = \frac{\sqrt{2}}{2\sqrt{y}} = \frac{1}{\sqrt{2y}}$$

$$ds = \sqrt{1 + \frac{1}{2y}}\, dy$$

$$SA = \int 2\pi x \, ds$$

$$= \int_{1/2}^{2} 2\pi \left(\sqrt{2y}\right) \sqrt{1 + \frac{1}{2y}}\, dy$$

$$= \int_{1/2}^{2} 2\pi \sqrt{2y\left(1 + \frac{1}{2y}\right)}\, dy$$

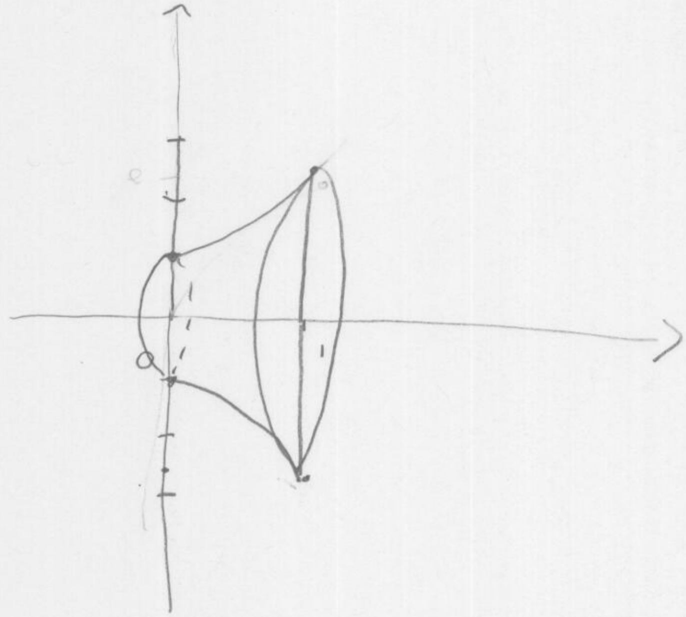
$$= 2\pi \int_{1/2}^{2} \sqrt{2y + 1}\, dy \qquad u = 2y + 1, \quad du = 2dy$$

$$= \pi \int_2^5 \sqrt{u}\, du$$

$$= \frac{2\pi}{3}\left[5^{3/2} - 2^{3/2}\right]$$

Example 3

Find the area of the surface generated by rotating $y = e^x$, $0 \le x \le 1$, about the x-axis



$$SA = \int 2\pi y \, ds$$

$$= \int_0^1 2\pi e^x \sqrt{1+e^{2x}} \, dx$$

$u = e^x$

$du = e^x dx$

$$= \int_1^e 2\pi \sqrt{1+u^2} \, du$$

$u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(e)} 2\pi \sqrt{1+\tan^2\theta} \sec^2\theta \, d\theta$$

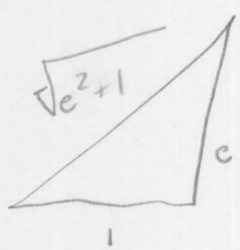
$$= 2\pi \int_{\frac{\pi}{4}}^{\tan^{-1} e} \sec^3\theta \, d\theta$$

$$= 2\pi \left[\tfrac{1}{2} (\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|) \right]_{\frac{\pi}{4}}^{\tan^{-1}(e)}$$

$$= \pi \left[\left(e\sqrt{1+e^2} + \ln|e + \sqrt{1+e^2}| \right) - \left(\sqrt{2}(1) - \ln|\sqrt{2}+1| \right) \right]$$

$$\frac{dy}{dx} = e^x$$

$$ds = \sqrt{1+e^{2x}} \, dx$$



Challenge: Arclength!