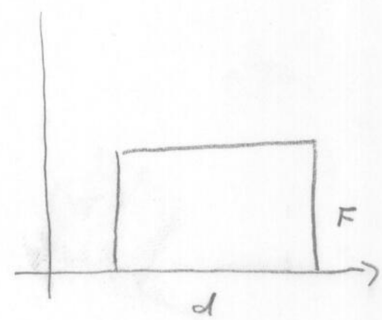
<u>Work</u> - <u>Lecture 8</u> (Cylindrical shells are in Lecture 7)

A physics application.

<u>Work</u> is effort needed to perform a task. If an object moves in a straight line with position $s(t)$, the force F on the object is given by

$$\left.\begin{aligned} F &= m \cdot s''(t) \\ &= m \cdot a(t) \end{aligned}\right] \text{Newton's Second Law}$$

Then when $a(t)$ is constant, $\text{Work} = F \cdot d$
$= \text{Force} \cdot \text{distance}$



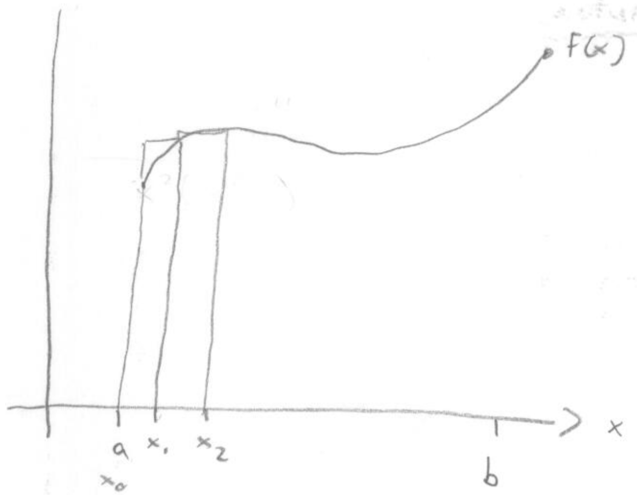
<u>Example</u> (Easy) If acceleration due to gravity is $-9.8\ m/s^2$, how much work is done by lifting a 2.1 kg book from the ground to a table .7 m high?

$$\begin{aligned} W &= F \times d \\ &= m \cdot a \cdot d \\ &= (2.1\ kg)(9.8\ m/s^2)(.7\ m) \\ &= -14.4\ N \cdot m \end{aligned}$$

$1\ N = 1\ kg \cdot m/s^2$

What if acceleration (& consequently force) varies?

Let force be a function of position.



Force $f(x)$ acting on an object from position a to position b. Divide into n subintervals w/ endpoints $x_0, \dots, x_n$. Pick x_i^* a sample point in $[x_{i-1}, x_i]$. For n large enough f is nearly constant on $[x_{i-1}, x_i]$.

Force as object travels x_{i-1} to x_i $\approx f(x_i^*)$

Work " " " " " $\approx f(x_i^*)\Delta x$

So total work $\approx \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Ergo

Defn The work done in moving an object from a to b is

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Example 1 When a particle is x m from the origin, a force of $x^2 + 3x$ N acts on it. How much work is done in moving from $x=2$ to $x=5$?

$$W = \int_2^5 [x^2 + 3x]\,dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 \Big]_2^5$$

$$= \left[\frac{125}{3} - \frac{75}{2}\right] - \left[\frac{8}{3} + \frac{12}{2}\right]$$

$$= \frac{117}{3} - \frac{63}{2}$$

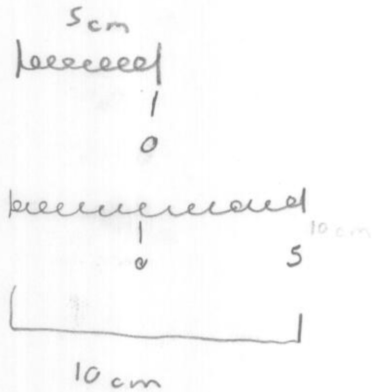
$$= 39 - \frac{63}{2}$$

$$= \frac{15}{2} \text{ N}\cdot\text{m}$$

Example 2 Hooke's Law states the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx \quad \text{for } k > 0.$$

A force of 30 N is needed to hold a spring stretched from its natural length of 5 cm to 10 cm. How much work is done in stretching from 10 to 13 cm?



$$f(x) = kx$$

$$30\text{ N} = k \cdot 5\text{ cm}$$

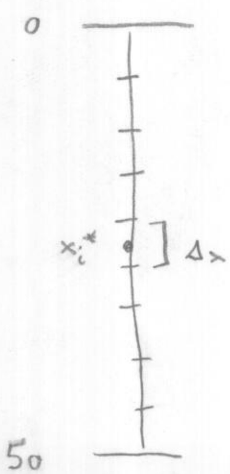
$$6\ \text{N/cm} = k$$

$$F(x) = 6x \text{ for } x \text{ in cm}$$

$$\int_{x=5}^{x=8} 6x\,dx = 3x^2 \Big|_5^8$$

$$= 117 \text{ N}\cdot\text{m}$$

Example 3 A 100-kg cable is 50 m long & hangs vertically from the top of a tall building. How much work to lift the cable to the top of the building?



Divide into n subintervals. Each masses $2\Delta x$ and is lifted approximately x_i^* for $x_i^* \in [x_{i-1}, x_i]$ a sample point.

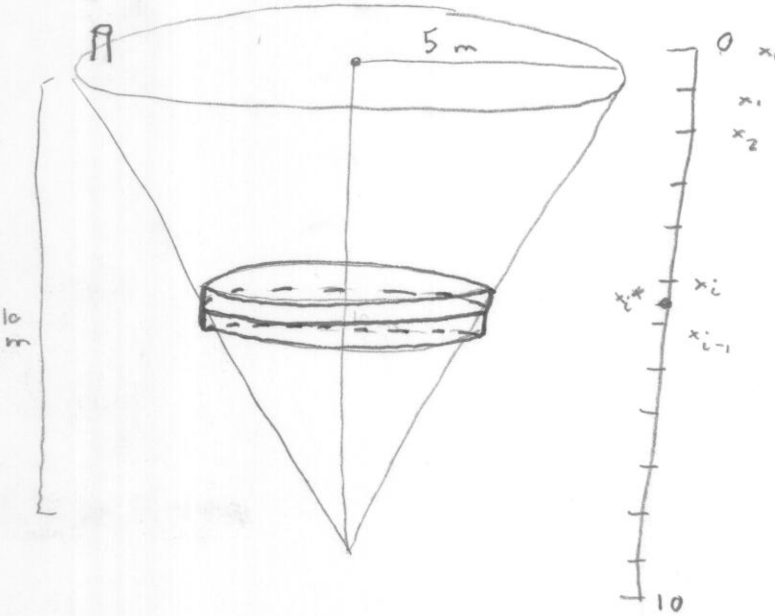
$$\text{Work} \sim \sum_{i=1}^{n} 2\Delta x(9.8) \cdot x_i^*$$

$$W = \int_0^{50} 19.6x\,dx = 9.8\,x^2 \Big|_0^{50}$$

$$= 9.8(2500) \text{ N}\cdot\text{m}$$

$$= 24{,}500 \text{ N}\cdot\text{m}$$

A conical tank of base radius 5 m and height 10 m is full of water (density 1000 kg/m^3). How much work is done by pumping all the water out of the top of the tank?



Divide vertical coordinate into n subintervals; n layers of water. Layer i is approximated by a cylinder of radius r_i^* and lifted a distance $\sim x_i^*$.

Work done on cylinder

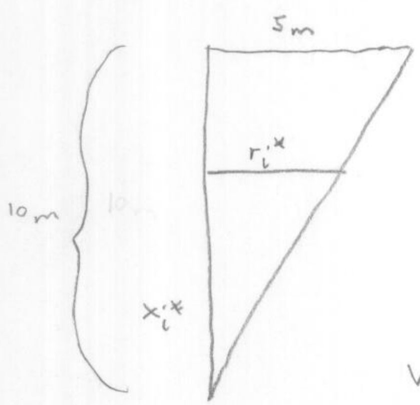
$$= (\text{Mass water}) \cdot \underset{\text{to gravity}}{\text{acceleration due}} \cdot \text{distance}$$

$$= (\text{Volume water}) \cdot 1000 \, kg/m^3 \, (9.8 \, m/s^2)(x_i^* \, m)$$

$$= [\pi {r_i^*}^2 \Delta x \; m^3]\left[9800 \frac{kg}{m^2 \cdot s^2}\right] x_i^*$$

$$= 9800\pi \, x_i^* \, {r_i^*}^2 \Delta x \quad N$$

Need r_i^*



$$\frac{5}{10} = \frac{r_i^*}{x_i^*}$$

$$\frac{1}{2} x_i^* = r_i^*$$

Work done on cylinder $= 9800\pi (x_i^*)\left(\frac{1}{2} x_i^*\right)^2 \Delta x \quad N$

$$= 2450\pi \, {x_i^*}^3 \Delta x \quad N$$

Total work $\approx \sum_{i=1}^{n} 2450\pi \, {x_i^*}^3 \Delta x \quad N$

$$\text{Work} = \int_0^{10} 2450\pi x^3 \, dx = 2450\pi \left(\frac{1}{4}x^4\right)\Big|_0^{10}$$

$$= 612{,}500 \; N \cdot m = 6.125 \times 10^5 \; N \cdot m$$