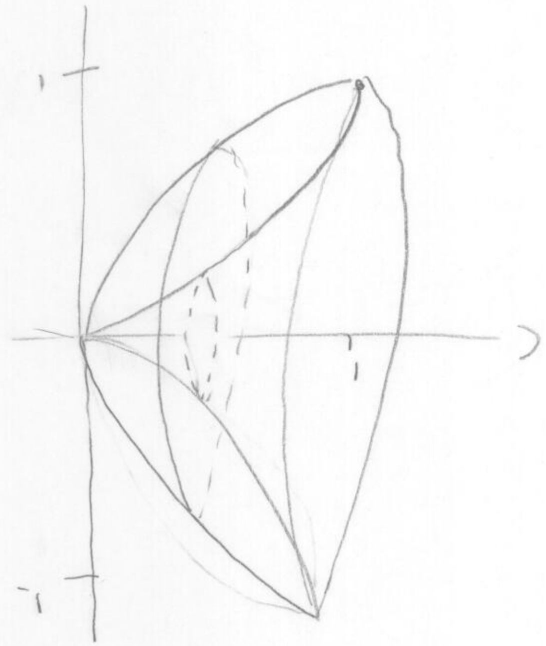
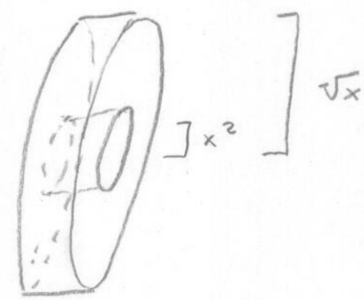
A more complex example

Example 3 The region R enclosed by curves $y = \sqrt{x}$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



A cylindrical cross-section is a circular washer



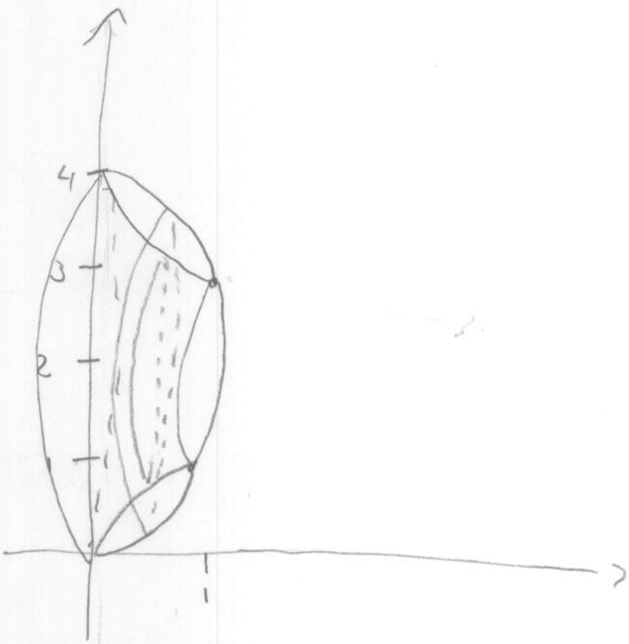
$$A(x) = \pi(\sqrt{x})^2 - \pi(x^2)^2$$
$$= \pi(x - x^4)$$

$$V = \int_0^1 \pi(x - x^4)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1$$
$$= \pi\left[\tfrac{1}{2} - \tfrac{1}{5}\right]$$
$$= \frac{3\pi}{10}$$

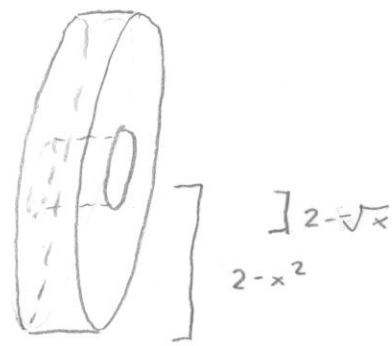
So if cross section is a washer w/ some inner and outer radius,

$$A = \pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right]$$

Example 4 Find the volume obtained by rotating the region in Example 3 about $y = 2$.



Cross section



$$A(x) = \pi\left[(2-x^2)^2 - (2-\sqrt{x})^2\right]$$

$$= \pi\left[4 - 4x^2 + x^4 - (4 - 4\sqrt{x} + x)\right]$$

$$= \pi\left[x^4 - 4x^2 + x + 4\sqrt{x}\right]$$

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left[x^4 - 4x^2 - x + 4\sqrt{x}\right]dx$$

$$= \pi\left[\frac{1}{5}x^5 - \frac{4}{3}x^3 - \frac{1}{2}x^2 + \frac{8}{3}x^{2/3}\right]_0^1$$

$$= \pi\left[\frac{1}{5} - \frac{4}{3} - \frac{1}{2} + \frac{8}{3}\right]$$

$$= \frac{31}{30}$$

Example 5

Volume of solid obtained by revolving (finite) region bounded by

$$\begin{cases} y = 1+\sec x \\ y = 3 \end{cases} \quad \text{about } y=1$$

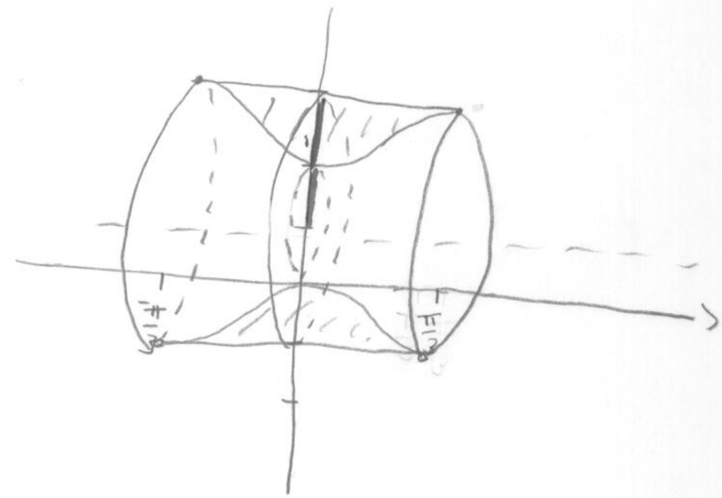
<u>Points of intersection</u>

$3 = 1+\sec x$

$2 = \sec x$

$\frac{1}{2} = \cos x$

$-\frac{\pi}{3}, \frac{\pi}{3} = x$



<u>Outer radius</u> 2

<u>Inner Radius</u> $\sec x$

$A(x) = \pi(2^2 - \sec^2 x)$

$$V = \int_{-\frac{\pi}{3}}^{\pi/3} \pi(4-\sec^2 x)\,dx$$

$$= 2\pi \int_0^{\frac{\pi}{3}} (4-\sec^2 x)\,dx$$

$$= 2\pi \left[4x - \tan x\right]_0^{\pi/3}$$

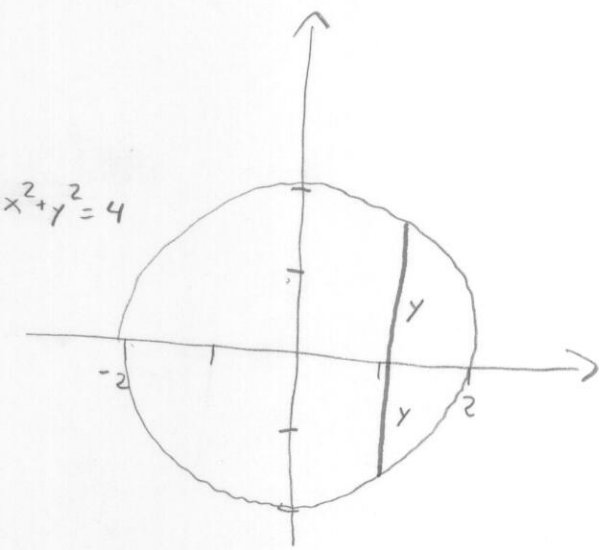
$$= 2\pi\left[\frac{4\pi}{3} - \sqrt{3}\right]$$

Quirky cross-sectional areas

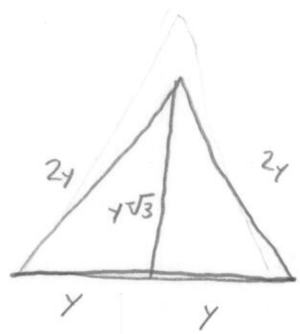
Suppose we don't have a solid of revolution but we do have some data on what the cross-sections of a shape look like. Then we can still make some headway.

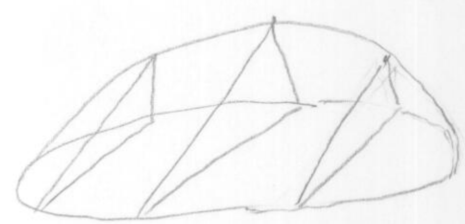
Example 1

Find the volume of a solid w/ base a circle of radius 2 and cross-sections $\perp$ to the base equilateral triangles.



Base of solid





So $A(x) = $ (area of triangle)

$$= \frac{1}{2}(2y)(\sqrt{3}y)$$

$$= \sqrt{3}y^2$$

$$= \sqrt{3}(4-x^2)$$

$$V = \int_{x=-2}^{x=2} \sqrt{3}(4-x^2)\,dx$$

$$= 2\sqrt{3}\int_0^2 (4-x^2)\,dx$$

$$= 2\sqrt{3}\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

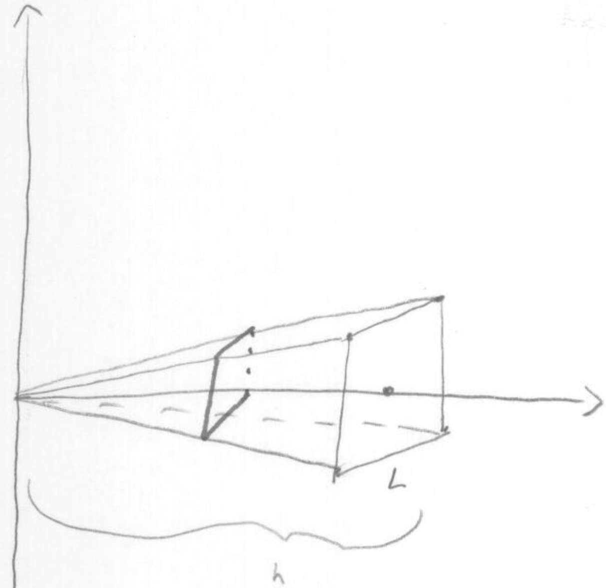
$$= 2\sqrt{3}\left(8 - \frac{8}{3}\right)$$

$$= \frac{32}{3}\sqrt{3}$$

Example 2

Find the volume of a pyramid whose base is a square w/ side L and whose height is h.

Doesn't come w/ coordinates, so we choose some.



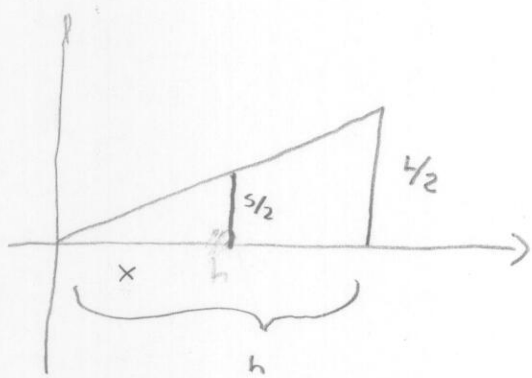
Cross sections are squares of side length s.

Looking at similar triangles in xy plane gives.

$$\frac{\frac{s}{2}}{x} = \frac{\frac{L}{2}}{h}$$

$$s = \frac{Lx}{h}$$

$$A(x) = \frac{L^2}{h^2}x^2$$



$$\text{Volume} = \int_{x=0}^{x=h} \frac{L^2}{h^2}x^2\,dx$$

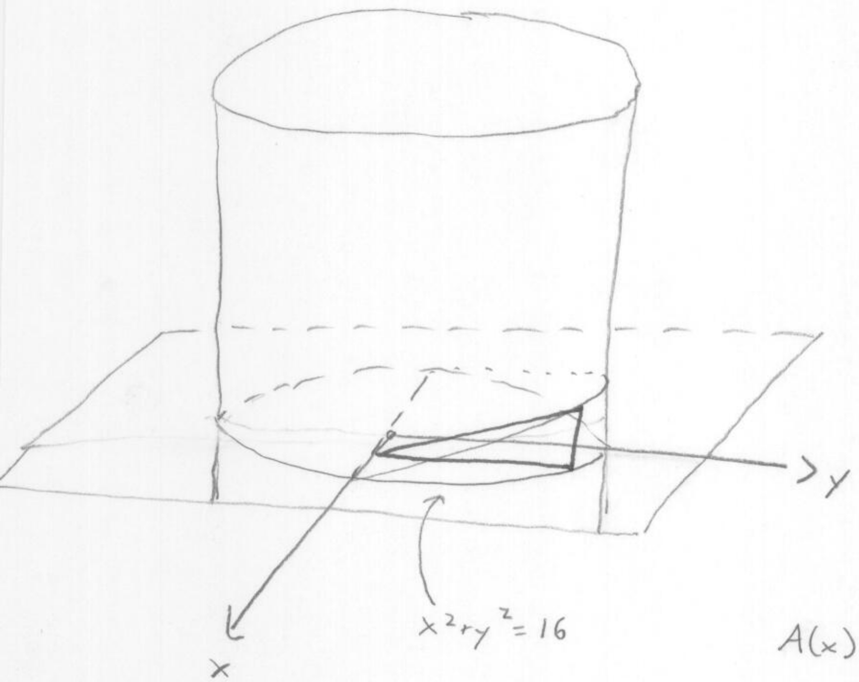
$$= \frac{L^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

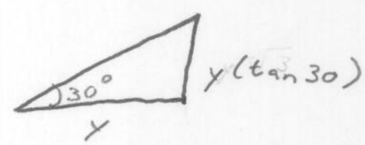
$$= \frac{L^2}{h^2}\left[\frac{1}{3}h^3\right]$$

$$= \frac{L^2h}{3}$$

Example 3 (More exciting!)

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is $\perp$ to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.





$$A(x) = \frac{1}{2}(y)\left(y \tan\frac{\pi}{3}\right)$$

$$= \frac{1}{2}y^2\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{16-x^2}{2\sqrt{3}}$$

$$V = \int_{-4}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx = \frac{2}{2\sqrt{3}}\int_0^4 (16-x^2)\,dx$$

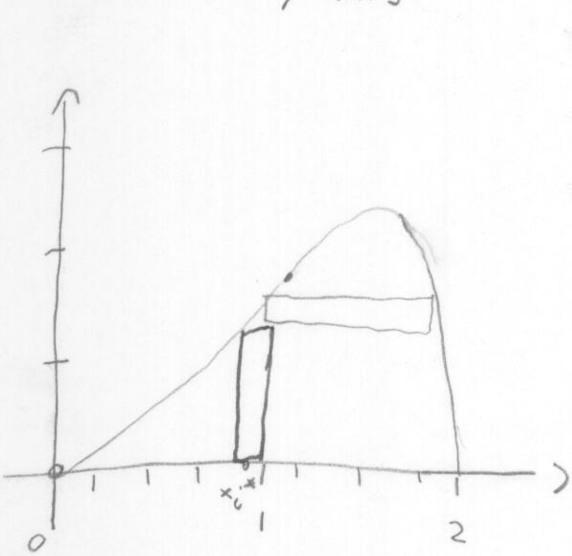
$$= \frac{1}{\sqrt{3}}\left[16x - \frac{1}{3}x^3\right]_0^4$$

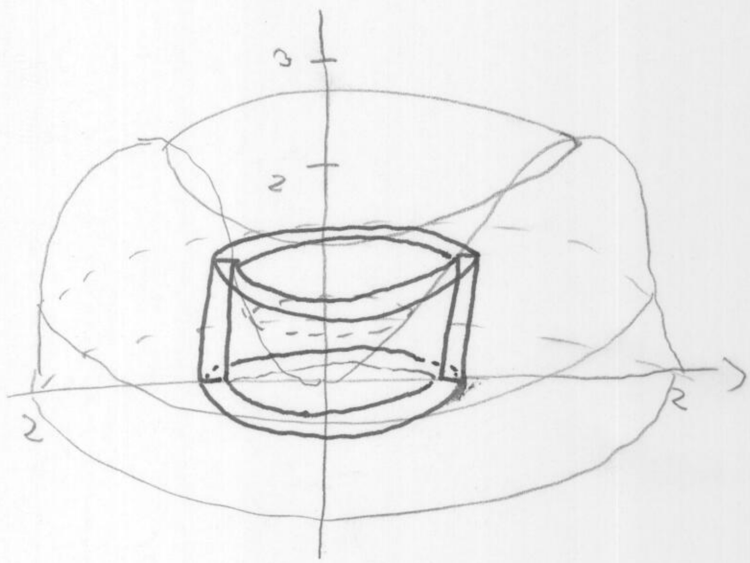
$$= \frac{1}{\sqrt{3}}\left[64 - \frac{64}{3}\right]$$

$$= \frac{128}{3\sqrt{3}}$$

Volumes by Cylindrical Shells

Problem Find the volume obtained by rotating region bounded by $y = 3x^2 - x^3$ and $y = 0$ about the y-axis





- Washer method is not great since we have to work out two strange functions of y, and also find maximum.

- Since we can't approximate by cross-sections, we instead approximate by cylindrical shells

- Take a rectangle parallel to the axis and rotate it around the axis

$$\text{Volume}_{\text{cylinder}} = V_{\text{outer cylinder}} - V_{\text{inner cylinder}}$$

$$= \pi r_1^2 h - \pi r_2^2 h$$

$$= \pi h [r_1^2 - r_2^2]$$

$$= \pi h [r_1 + r_2][r_1 - r_2]$$

$$= 2\pi h \left[\frac{r_1 + r_2}{2}\right] \Delta r$$

Average radius of shell

e.g. Volume = "circumference" · "thickness"

In our example: $r = x_i^*$

$h = f(x_i^*)$

$\Delta r = \Delta x$

$$\text{Volume}_{\text{cylindrical shell}} = 2\pi\,(f(x_i^*))\,x_i^*\,\Delta x$$

$$\text{Volume}_{\text{solid}} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\, f(x_i^*)\, x_i^*\, \Delta x$$

$$= \int_0^2 2\pi\, f(x)\, x\, dx$$

$$= 2\pi \int_0^2 (3x^2 - x^3)\, x\, dx$$

$$= 2\pi \left[\frac{3}{4}x^4 - \frac{1}{5}x^5 \right]_0^3$$

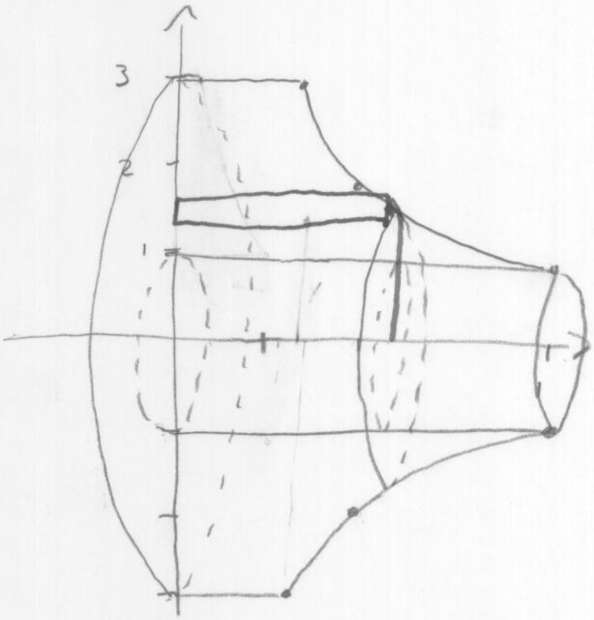
$$= 2\pi \left[\frac{243}{4} - \frac{243}{5} \right]$$

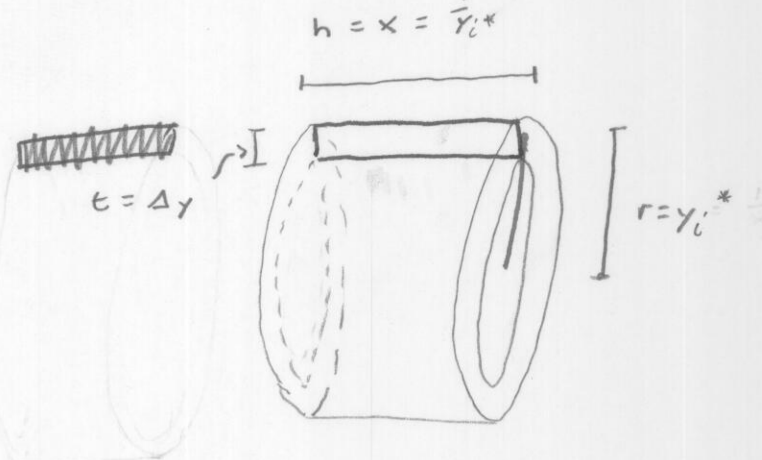
$$= 2\pi \left[\frac{243}{20} \right]$$

$$= \frac{243\pi}{10}$$

So we can approximate a volume by taking a rectangle <u>parallel</u> to the axis of rotation, using it to generate a cylindrical shell whose volume is $2\pi \underbrace{r}_{\text{distance to axis}}\ \underbrace{h}_{\text{height of rectangle}}\ \underbrace{t}_{\text{thickness of rectangle (usually } dx \text{ or } dy)}$

② Find the volume of the solid obtained by rotating the region bounded by $xy = 1$, $x = 0$, $y = 1$, and $y = 3$ about the x-axis.





$$V_{cylinder} = 2\pi r h t$$

$$= 2\pi \left[y_i^*\right] \frac{1}{y_i^*} \Delta y$$

$$\text{Volume} = \lim_{n \to \infty} \sum_{i=0}^{n} 2\pi y_i^* \frac{1}{y_i^*} \Delta y$$

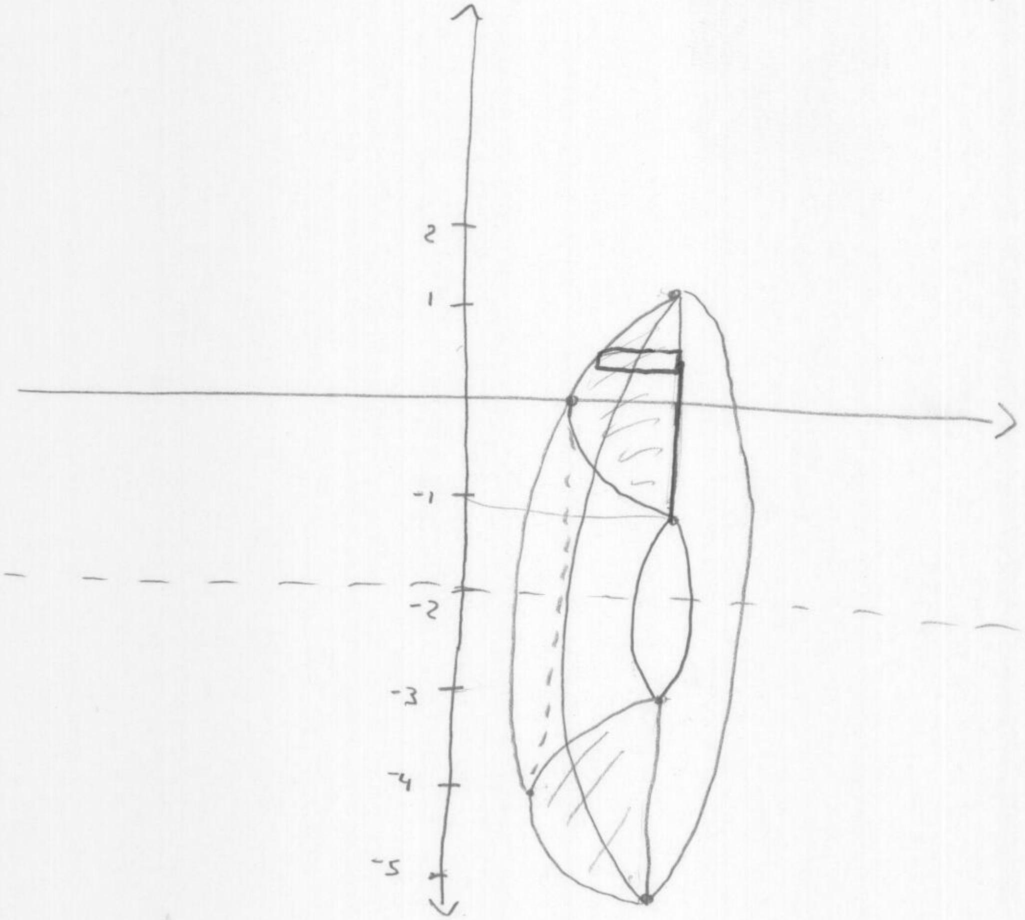
$$= \int_{y=1}^{y=3} 2\pi \, dy$$

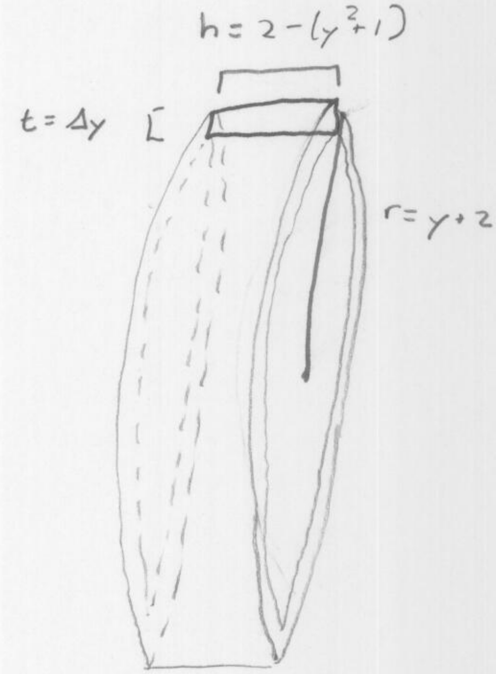
$$= 2\pi \left[3 - 1\right]$$

$$= 4\pi$$

Examples

(3) Find the volume of the solid generated by rotating the region bounded by $x = y^2+1$ and $x=2$ about $y=-2$.





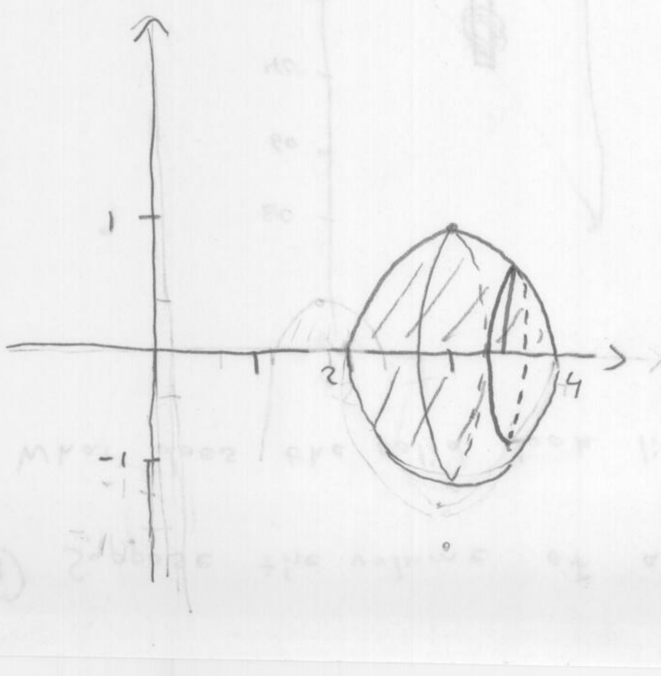
$$\text{Volume}_{\text{cylinder}} = 2\pi h r t$$
$$= 2\pi[1-y^2][y+2]\,\Delta y$$

$$\text{Volume} = \int_{y=-1}^{y=1} 2\pi(1-y^2)(y+2)\,dy$$
$$= 2\pi \int_{-1}^{1} [y+2-y^3-2y^2]\,dy$$
$$= 2\pi(2)\int_0^1 [2-2y^2]\,dy$$
$$= 4\pi\left[2y - \frac{2}{3}y^3\right]_0^1$$
$$= 4\pi\left[2-\frac{2}{3}\right]$$
$$= \frac{16\pi}{3}$$

Volumes of solids

General	Place on axis, find cross-sectional area
Solids of Revolution	**Disk/washer** Approximate by spinning a rectangle perpendicular to axis of rotation $V = \int \pi [R^2 - r^2]\, dx$ $V = \int \pi [R^2 - r^2]\, dy$
	Cylindrical shells Approximate by spinning a rectangle parallel to axis of rotation $V = \int 2\pi r h\, \Delta x$ $V = \int 2\pi r h\, \Delta y$

Example 4 Set up integrals to find the volume of the solids given by rotating the region bounded by $y = -x^2 + 6x - 8$ and $y = 0$ about the x-axis and y-axis.



$0 = -x^2 + 6x - 8$

$0 = x^2 - 6x + 8$

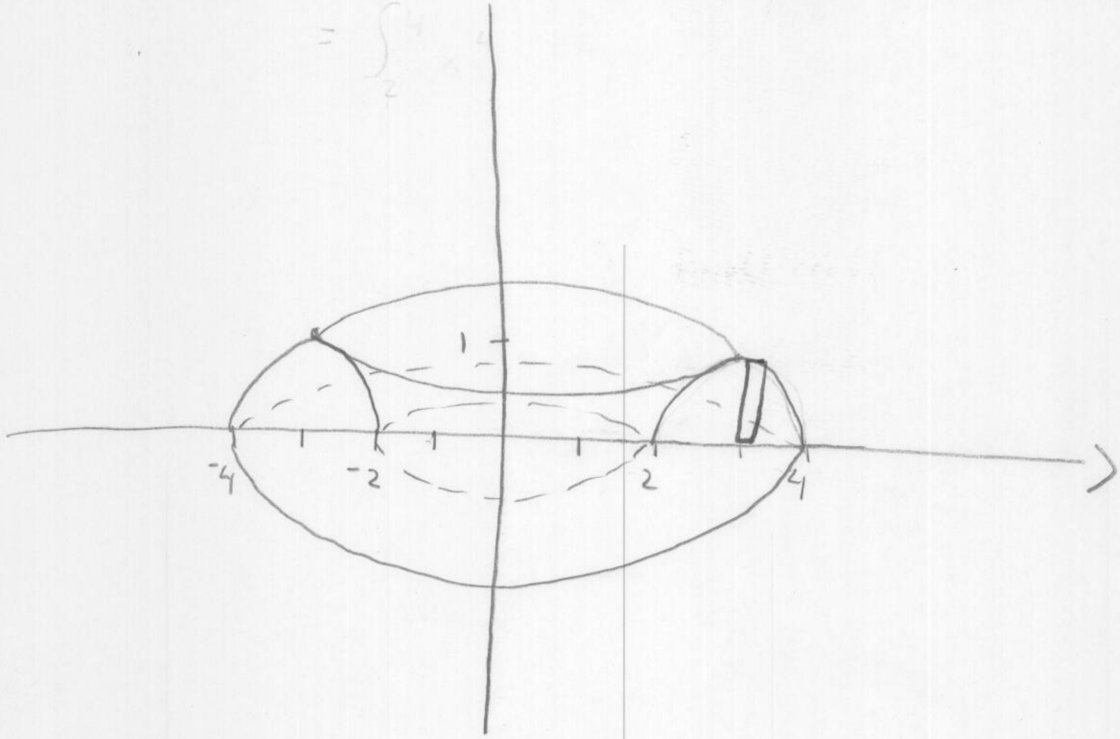
$(x-4)(x-2)$

Vertex at 3

$-9 + 18 - 8 = 1$

x-axis: Disk method $r = y = -x^2 + 6x - 8$

$$V = \int_2^4 [-x^2 + 6x - 8]^2\, dx$$



y-axis: cylindrical shells $r = x$ $h = y = -x^2+6x-8$

$$V = \int_2^4 2\pi x\left[-x^2+6x-8\right]dx$$