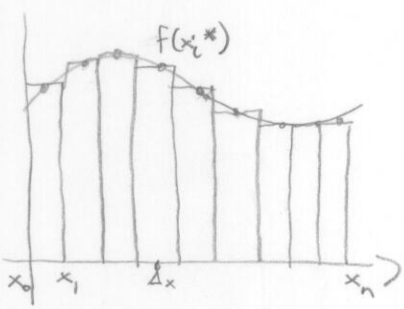
Lecture 6

Applications of Integration

- Average Value of a Function
- Areas Between Curves
- Volumes
- Work

Average value of a Function

Average of a finite collection $a_1, \dots, a_n$ is $\dfrac{a_1 + \dots + a_n}{n}$



$\Delta x = \dfrac{b-a}{n}$

$$\text{Average value of a function} \approx \frac{f(x_1^*) + \dots + f(x_n^*)}{n}$$

$$= \left(\frac{1}{b-a}\right)\left(\frac{b-a}{n}\right) f(x_1^*) + \dots + f(x_n^*)$$

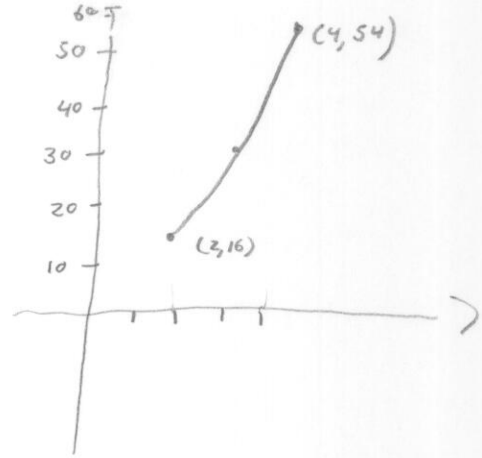
$$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Take a limit as $n \to \infty$

$$\text{Average value of a function} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \frac{1}{b-a} \int_a^b f(x)\,dx$$

Example 1 Average value of $f(x) = 3x^2 + x + 2$ on $[2, 4]$



$$\frac{1}{4-2}\int_2^4 (3x^2+x+2)\,dx = \frac{1}{2}\left[x^3 + \frac{1}{2}x^2 + 2x\right]_2^4$$

$$= \frac{1}{2}\left[(64+8+8) - (8+2+4)\right]$$

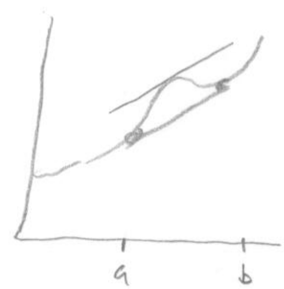
$$= \frac{1}{2}(66)$$

$$= 33$$

Mean Value Thm For Integrals

Recall If F diff'ble on (a,b), cts on $[a,b]$, $\exists\ c \in (a,b)$ st

$$F'(c) = \frac{F(b) - F(a)}{b-a}$$



If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$. So $\exists\ c$ st

$$g'(c) = \frac{g(b) - g(a)}{b-a}.$$

Thm

If f is cts on $[a,b]$, $\exists\ c \in (a,b)$ st

$$f(c) = \frac{\int_a^b f(t)\,dt}{b-a} = \frac{1}{b-a}\int_a^b f(t)\,dt = \text{Average value of } f \text{ on } [a,b]$$

That is, the average value is attained by some c.

Example 2 $f(x) = 3x^2 + x + 2$

Find c st $f(c) = \frac{1}{4-2}\int_2^4 f(x)\,dx$

$$f(c) = 33$$

$$3c^2 + c + 2 = 33$$

$$3c^2 + c - 31 = 0$$

$$c = \frac{-1 \pm \sqrt{1 - 4(3)(-31)}}{6}$$

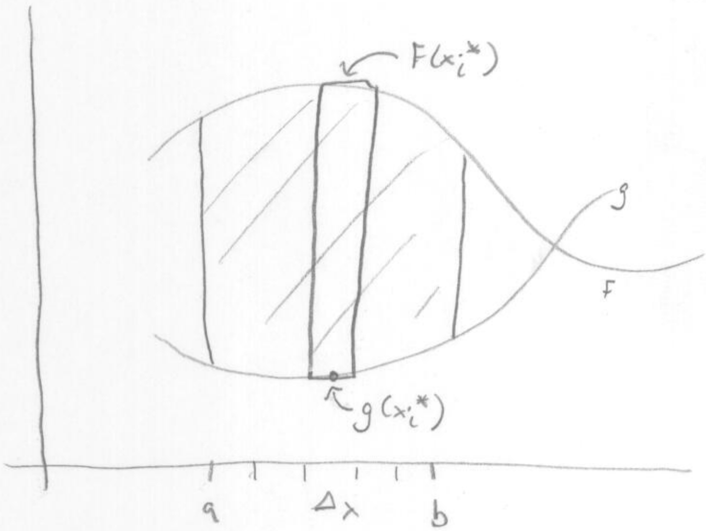
$$= \frac{-1 \pm \sqrt{372}}{6}$$

$\boxed{c = 3.04}$ $\quad$ $c = -3.38$ ✗

Areas Between Curves

So Far we've only discussed area under a curve, or between the curve and y-axis.

Today Areas between curves, and of planar regions.



Area is approximated by $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$. So we define

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

This is exactly the integral of $f-g$ from a to b.

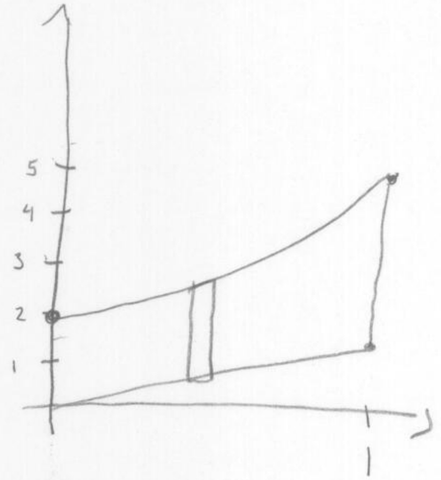
Thm The area A of the region bounded by $y = f(x)$, $y = g(x)$ and the lines $x=a$, $x=b$ for f, g cts and $f(x) \geq g(x)$ on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Examples

① Find the area bounded above by $y = 2e^x$, bounded below by $y = \frac{3}{2}x$, and at the sides by $x = 0$ & $x = 1$.

Step 1 Sketch region & an approximating rectangle.



Step Two

Identify top & bottom curves & write down integral.

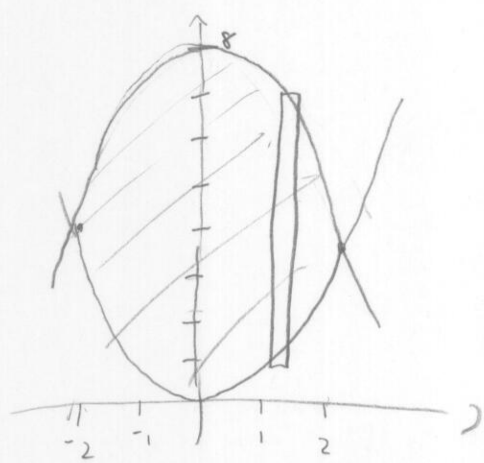
$$A = \int_0^1 \left[2e^x - \frac{3}{2}x\right]dx$$

$$= \left[2e^x - \frac{3}{4}x^2\right]_0^1$$

$$= \left[2e - \frac{3}{4}\right] - \left[2 - 0\right]$$

$$= 2e - \frac{11}{4}$$

(2) Find the area of a region enclosed by $y = x^2$ and $y = 8 - x^2$



Step One Sketch

Step Two Find points of intersection by solving $f(x) = g(x)$

$$x^2 = 8 - x^2$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2$$

$$A = \int_{-2}^{2} \left[(8 - x^2) - x^2\right]dx$$

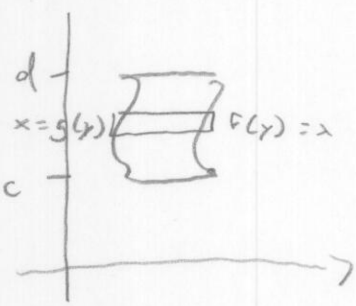
$$= 2\int_0^2 (8 - 2x^2)\,dx$$

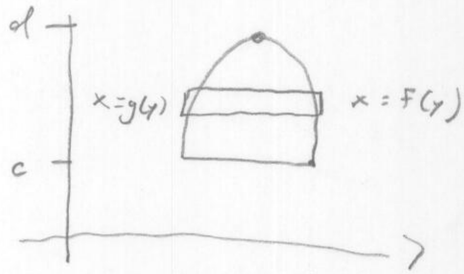
$$= 2\left[8x - \frac{2}{3}x^3\right]_0^2$$

$$= 2\left[16 - \frac{16}{3}\right]$$

$$= \frac{64}{3}$$

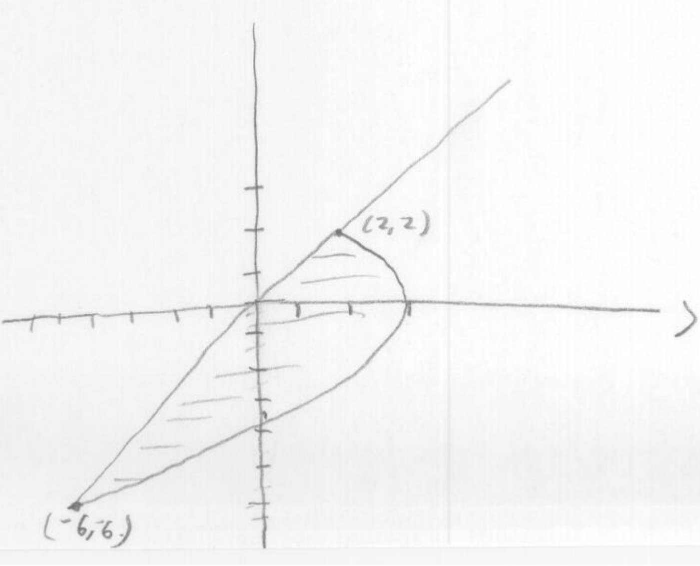
We see we sometimes need to determine the appropriate bounds of integration to find the area of a region. In fact, it can be slightly more complicated: not every region is best described in terms of functions of x.





Compute this area as $A = \int_c^d ((F(y) - g(y))\,dy$

Example Find the area bounded by $y = x$ and the parabola $4x + y^2 = 12$.



$x = y \qquad x = \frac{1}{4}(12 - y^2)$

$$y = \frac{1}{4}(12 - y^2)$$

$$4y = 12 - y^2$$

$$y^2 + 4y - 12 = 0$$

$$(y+6)(y-2)$$

$$y = -6 \quad y = 2$$

$(-6, -6) \quad (2, 2)$ } Intersection points

$$A = \int_{-6}^{2} \left[\tfrac{1}{4}(12 - y^2 - y)\right] dy$$

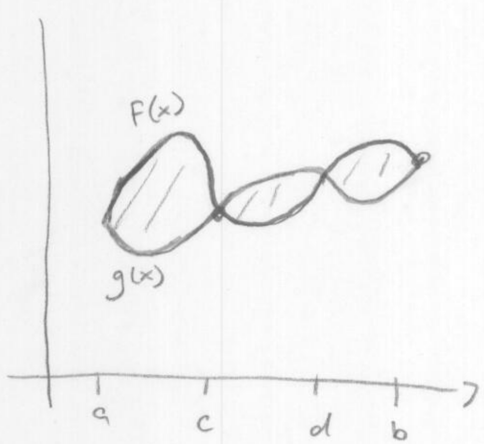
$$= \int_{-6}^{2} \left(3 - \tfrac{1}{4}y^2 - y\right) dy$$

$$= \left[3y - \tfrac{1}{12}y^3 - \tfrac{1}{2}y^2\right]_{-6}^{2}$$

$$= \left[6 - \tfrac{1}{3} - 2\right] - \left[-18 - (-18) - 18\right]$$

$$= \frac{65}{3}$$

So far we've dealt with regions in which $f(x) \geq g(x)$ on $[a, b]$ or $f(y) \geq g(y)$ on $[c, d]$. What about curves that cross each other?



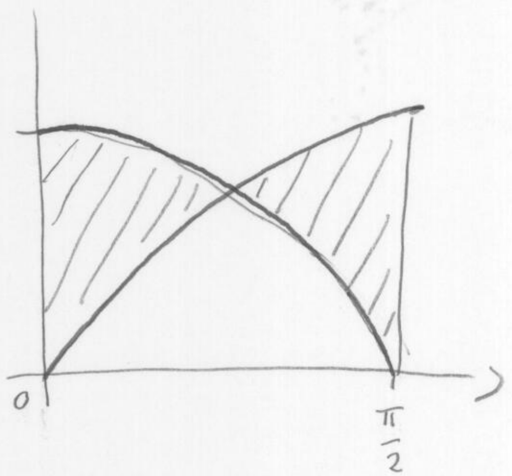
$$A = \int_a^c [F(x) - g(x)]\,dx + \int_c^d [g(x) - F(x)]\,dx + \int_d^b [F(x) - g(x)]\,dx$$

$$= \int_a^b |F(x) - g(x)|\,dx$$

Have to break the interval $[a, b]$ onto subintervals on which $F \geq g$ or $g \geq F$.

Example

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.



Solve for intersection points

$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$$

$$A = \int_0^{\pi/4} [\cos x - \sin x]\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x]\,dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{+1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (-0-1)\right] + \left[(0+1) - \left(\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right)\right]$$

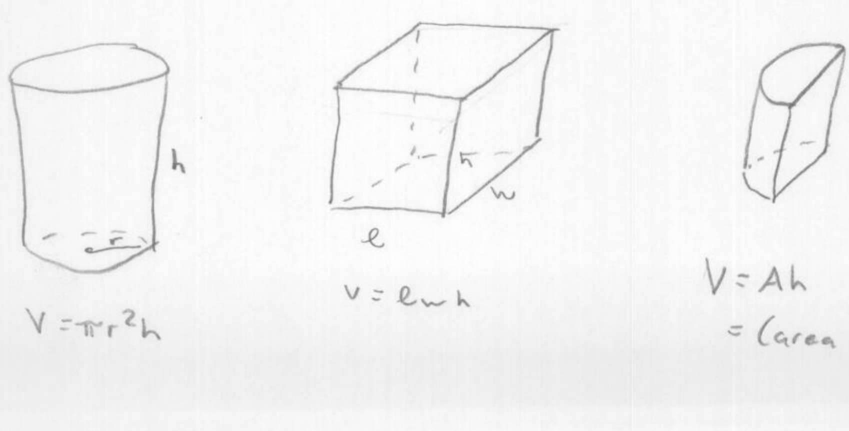
$$= -2 + \frac{4}{\sqrt{2}}$$

$$= -2 + 2\sqrt{2}$$

Volume

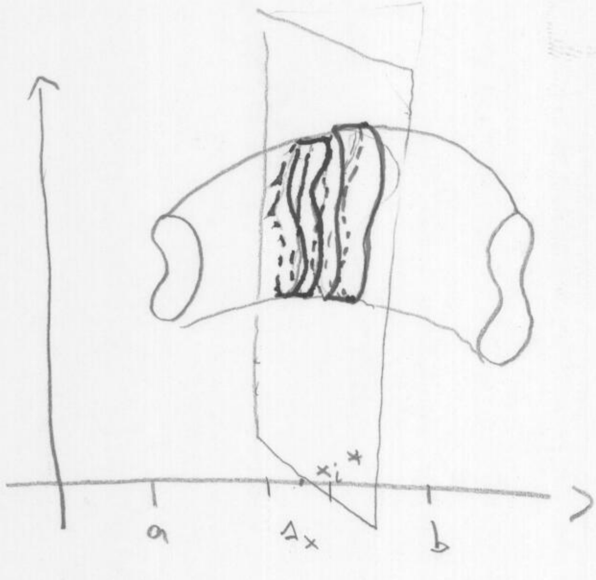
We need to give a precise definition of volume in the way we did for area.

Case I Cylinders



$V = \pi r^2 h$ $\quad$ $V = lwh$ $\quad$ $V = Ah = (\text{area of base})\,\text{height}$

In the same way that approximation by <u>rectangles</u> was how we defined area, we define volume by approximation by <u>cylinders</u>.



Area of cross-section:

$A(x)$ Function of x

$$\text{Volume}_{\text{cylinder}} = \Delta x \cdot A(x_i^*)$$

$$\text{Volume}_{\text{total}} \approx \sum_{i=1}^{n} \Delta x \, A(x_i^*)$$

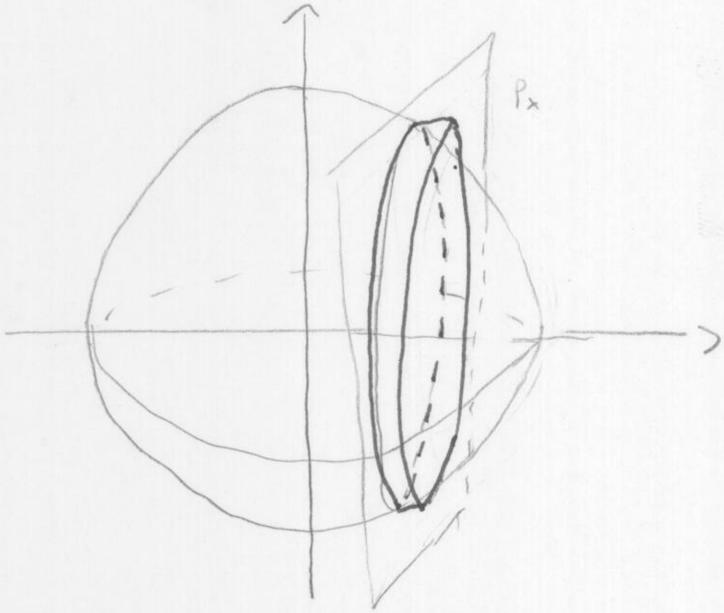
<u>Defn</u> Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in the plane P_x through x & perpendicular to the x-axis, is a cts ftn $A(x)$, the <u>volume</u> of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

$$= \int_a^b A(x)\,dx$$

Let's get an algorithm for computing A cross-sectional area in a special case.

Example Sphere of radius 1



Cross sections $A(x)$ are circles with radius $y = \sqrt{1-x^2}$. So in fact

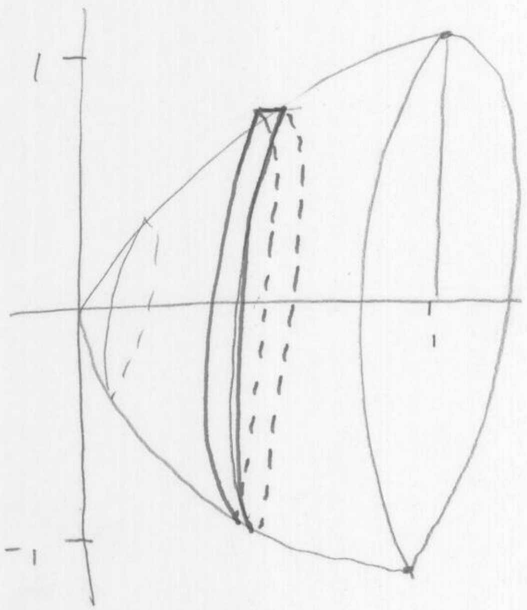
$$A(x) = \pi\left(\sqrt{1-x^2}\right)^2 = \pi(1-x^2)$$

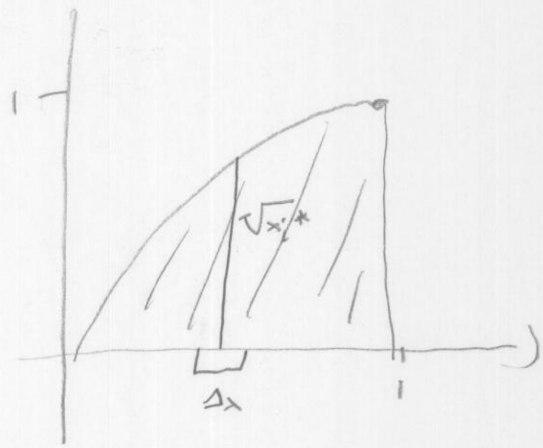
$$\text{Volume} = \int_{-1}^{1} \pi(1-x^2)\,dx = 2\int_0^1 \pi(1-x^2)\,dx = 2\pi\left[x - \tfrac{1}{3}x^3\right]_0^1 = 2\pi\left(1-\tfrac{1}{3}\right) = \tfrac{4}{3}\pi, \text{ as expected}$$

So we know how to find the volume of solids if we can determine cross-sectional area. We'll discuss further cases of applying this to individual solids shortly, but first let's study a large class of solids called solids of revolution for which cross-sectional area is simple to compute.

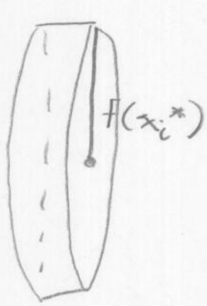
Solids of Revolution

Example Find the volume of a solid by rotating about the axis the region under $y = \sqrt{x}$ from 0 to 1.





Break $[0,1]$ into n subintervals w/ endpoints $x_0, x_1, \ldots, x_n$. Approximate solid by n cylinders



w/ height Δx; radius $f(x_i^*) = \sqrt{x_i^*} = r_i^*$

$$\text{Volume single cylinder} = A(x_i^*)\,\Delta x$$
$$= \pi\left(\sqrt{x_i^*}\right)^2 \cdot \Delta x$$
$$= \pi x_i^* \,\Delta x$$

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \pi x_i^* \,\Delta x$$

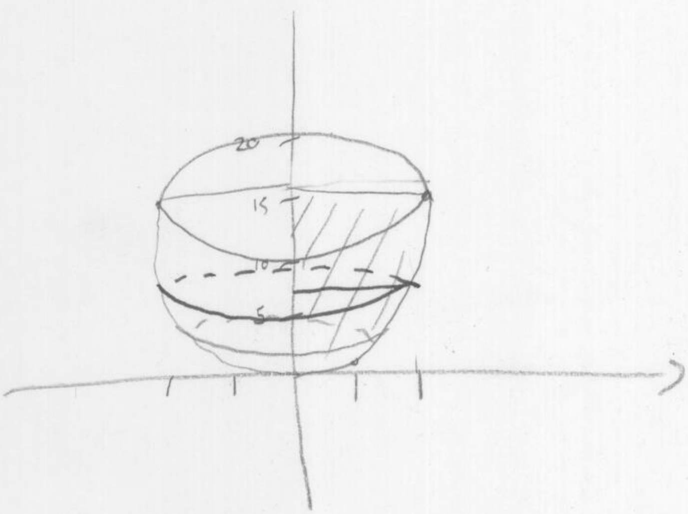
$$V = \int_0^1 \pi x \, dx = \frac{\pi}{2} x^2 \Big|_0^1$$
$$= \frac{\pi}{2}$$

The point Cross-sections were circular solids w/ known radius (y in this case. Then $A(x) = \pi r^2$, and $V = \int \pi r^2 dx$.

Of course this won't always be wrt x.

Example 2

Find the volume of a solid obtained by rotating the region bounded by $y = x^4$, $y = 16$, and $x = 0$ about the y-axis.



Circular cross sections with radius $x = y^{1/4}$

$$A(y) = \pi r^2$$
$$= \pi (y^{\frac{1}{4}})^2$$
$$= \pi y^{1/2}$$

$$V = \int_0^{16} \pi y^{1/2} dy$$
$$= \pi \left(\tfrac{2}{3} y^{3/2}\right)\Big|_0^{16}$$
$$= \frac{2\pi}{3}(64)$$
$$= \frac{128\pi}{3}$$

So the volume of a solid is defined by $V = \int_a^b A(x)dx$ or $V = \int_a^b A(y)dy$ and if the cross-sectional area is a circle, $A = \pi (\text{radius})^2$.