Integration of Rational Functions

$\int \frac{dx}{x^2+5x+6} = ?$ Could do a trig substitution... but there's a better option.

Motivation: write as two linear factors

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

$1 = A(x+3) + B(x+2)$ valid for all x

$x=-3$ $\quad 1 = B(-3+2)$

$1 = -B$

$B = -1$

$x = -2$ $\quad 1 = A(-2+3)$

$1 = A$

$$\frac{1}{x^2+5x+6} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int \frac{dx}{x^2+5x+6} = \int \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx$$

$$= \ln|x+2| - \ln|x+3| + C$$

We can use this method whenever the integrand has a denominator which is a product of linear factors.

In general, when $f(x) = \frac{P(x)}{Q(x)}$ and $\deg P < \deg Q$, we can express as a product of simpler fractions,

When $\deg P > \deg Q$, we can long divide until $f(x) = S(x) + \frac{R(x)}{Q(x)}$ where $R(x)$ is the remainder.

$$\int \frac{x^3 + x^2 + x}{x - 1}\, dx$$

$$\begin{array}{r}
x^2 + 2x + 3 \\
x - 1 \,\big)\overline{\, x^3 + x^2 + x + 0} \\
-(x^3 - x^2) \\
\hline
2x^2 + x + 0 \\
-(2x^2 - 2x) \\
\hline
3x + 0 \\
-(3x - 3) \\
\hline
3
\end{array}$$

$$= \int \left(x^2 + 2x + 3 + \frac{3}{x-1}\right) dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x + 3\ln|x-1| + C$$

- Step 1 Long Divide $f(x) = S(x) + \frac{R(x)}{Q(x)}$
- Step 2 Write as a sum of terms $\frac{A}{(ax+b)^i}$ $\frac{Ax+B}{(ax^2+bx+c)^j}$

Case I $Q(x)$ is a product of linear factors

$$Q(x) = (a_1x + b_1)\cdots(a_nx + b_n)$$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

Example

$$\int \frac{x^2+3x+1}{2x^3+5x^2+3x}\,dx = \int \frac{x^2+3x+1}{x(2x+3)(x+1)}\,dx$$

$x(2x^2+5x+3)$

$x(2x+3)(x+1)$

$$\frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x+1} = \frac{x^2+3x+1}{x(2x+3)(x+1)}$$

$$A(2x+3)(x+1) + Bx(x+1) + Cx(2x+3) = x^2+3x-1$$

$x=0$

$A(3)(1) = +1$

$A = \frac{+1}{3}$

$x = \frac{-3}{2}$ $\quad B\left(\frac{-3}{2}\right)\left(\frac{-3}{2}+1\right) = \frac{+9}{4} + \frac{-9}{2} + 1$ $\quad -18-4$

$B\left(\frac{3}{4}\right) = \frac{-23}{4}$

$B = \frac{-5}{4}$

$x=-1$ $\quad C(-1)(1) = 1-3+1$

$-C = -1$

$C = 1$

$$\int \frac{x^2+3x+1}{2x^3+3x^2+3x}\,dx = \int\left(\frac{-1}{3x} + \frac{-5/4}{2x+3} + \frac{1}{x+1}\right)dx$$

$$= \frac{-1}{3}\ln|x| - \frac{5}{4}\left(\frac{1}{2}\right)\ln|2x+3| + \ln|x+1| + C$$

Case II $Q(x)$ is a product of linear factors with some repeated.

$$Q(x) = (a_1x+b_1)^{k_1} \cdots (a_nx+b_n)^{k_n}$$

Then
$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \dots + \frac{A_{k_1}}{(a_1x+b_1)^{k_1}} + \dots + \frac{A_r}{(a_nx+b_n)^{k_n}}$$

Example

$$\int \frac{6x}{x^3-x^2-x+1}\,dx = \int \frac{6x}{(x-1)^2(x+1)}\,dx$$

$(x-1)(x^2-1)$

$(x-1)(x-1)(x+1)$

$(x-1)^2(x+1)$

$$\frac{6x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$6x = A(x-1)\cancel{(x+1)} + B(x+1) + C(x-1)^2$$

$x=1$

$$6 = 2B$$
$$3 = B$$

$x=-1$

$$-6 = C(-2)^2$$
$$\frac{-3}{2} = C$$

$x=0$

$$0 = -A + 3 + \frac{-3}{2}$$
$$\frac{-3}{2} = -A$$
$$\frac{3}{2} = A$$

$$\int \frac{6x}{x^3-x^2-x+1}\,dx = \int \left[\frac{3/2}{x-1} + \frac{3}{(x-1)^2} + \frac{-3/2}{x+1}\right] dx$$

$$= \frac{3}{2}\ln|x-1| + \frac{-3}{(x-1)} - \frac{3}{2}\ln|x+1| + c$$

Case III $Q(x)$ contains irreducible quadratic factors, not repeated

$$Q(x) = (a_1x+b_1)\cdots(a_nx+b_n)(a_{n+1}x^2+b_{n+1}x+c_{n+1})\cdots(a_mx^2+b_mx+c_m)$$

$\frac{P(x)}{Q(x)}$ includes a term $\frac{A_ex+B_e}{a_ex^2+b_ex+c}$ for each such quadratic factor

Example
$$\int \frac{3x^2-2x+9}{x^3+9x}\,dx \qquad \frac{Ax+B}{x^2+9} + \frac{C}{x} = \frac{3x^2-2x+9}{x^3+9x}$$

$x(x^2+9)$
↑
irreducible!

$$(Ax+B)x + C(x^2+9) = 3x^2-2x+9$$

$$Ax^2+Bx+Cx^2+9C = 3x^2-2x+9$$

$$(A+C)x^2+Bx+9C = 3x^2-2x+9$$

$$9C = 9$$
$$C = 1$$

$$B = -2$$

$$A+C = 3$$
$$A = 2$$

$$\int \frac{3x^2-2x+9}{x^3+9x}\,dx = \int\left[\frac{2x-2}{x^2+9} + \frac{1}{x}\right]dx$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

$$= \int\left(\frac{2x}{x^2+9} - \frac{2}{x^2+9}\right)dx + \ln|x| + C$$

$$= \int\left[\frac{2x}{x^2+9}dx - \frac{2/9}{\left(\frac{x}{3}\right)^2+1}\right]dx + \ln|x| + C$$

$$= \ln|x^2+9| - \frac{2}{9}\arctan\left(\frac{x}{3}\right) + \ln|x| + C$$

Case IV

$Q(x)$ contains a repeated quadratic factor

$$(ax^2+bx+c)^r \rightsquigarrow \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Example

$$\int \frac{1-2x+3x^2-x^3}{x(x^2+1)^2}\,dx$$

$$\frac{1-2x+3x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1-2x+3x^2-x^3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$x=0 \qquad 1 = A$

$$1-2x+3x^2-x^3 = x^4+2x^2+1 + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

$$= (B+1)x^4 + Cx^3 + (2+B+D)x^2 + (C+E)x + 1$$

$B = -1$

$C = -1$

$D = 2$

$2+B+D = 3 = -2$

$B+D = 1 = -3$

$D = 2$

$-2 = C+E$

$-1 = E$

$$= \int \left[\frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{2x-1}{(x^2+1)^2} \right] dx$$

$\sec^4 x\,dx$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \ln|x| - \arctan x + \int \frac{2x}{(x^2+1)^2}\,dx + 3\int \frac{1}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \arctan x + \frac{-1}{x^2+1} + 3\int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$-3\int \cos^2\theta\,d\theta$$

$$= \ln|x| - \arctan x - \frac{1}{x^2+1} - 3\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right] + C$$

$\sqrt{1+u^2}$ 1 $\sin\theta\cos\theta$

$$= \ln|x| - \arctan x - \frac{1}{x^2+1} - \frac{3}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + C$$

$$= \ln|x| - \frac{5}{2}\arctan x - \frac{\frac{3}{2}x + 1}{x^2+1} + C$$

Note Not always the easiest way

$$\int \frac{4x^3+21x^2+18x}{x^4+7x^3+9x^2+1}\,dx = \ln|x^4+7x^3+9x^2+1| + C$$

Example Can Use substitution to produce a partial Fraction

$$\int \frac{\sqrt{x+9}}{x}\,dx$$

$u = \sqrt{x+9}$ $\qquad du = \frac{1}{2\sqrt{x+9}}\,dx$

$u^2 = x+9$

$u^2 - 9 = x$ $\qquad 2u\,du = dx$

$$\int \frac{u}{u^2-9}\,2u\,du = \frac{1}{2}\int\left(\frac{1}{u+3} + \frac{1}{u-3}\right)du$$

$$\frac{u}{u^2-9} = \frac{A}{u+3} + \frac{B}{u-3}$$

$$= \frac{1}{2}\left(\ln|u+3| + \ln|u-3|\right) + C$$

$u = (u-3)A + (u+3)B$

$= \cancel{\ln}\ \frac{1}{2}$

$u = 3$ $\qquad 3 = 6B$

$\frac{1}{2} = B$

$u = -3$ $\qquad -3 = -6A$

$\frac{1}{2} = A$

$$\int \frac{u}{u^2-9}\, 2u\,du = \int \frac{2u^2}{(u^2-9)}\,du$$

$$\begin{array}{r} 2 \\ u^2-9\;\overline{)\,2u^2+0u+0} \\ -(2u^2+0u-18) \\ \hline 18 \end{array}$$

$$= \int 2 + \frac{18}{u^2-9}\,du$$

$$= 2u + 9\int\left[\frac{-1}{u+3} + \frac{1}{u-3}\right]du$$

$$= 2u + 9\ln|u+3| + 9\ln|u-3| + c$$

$$= 2u + 9\ln|u^2-9| + c$$

$$= 2\sqrt{x+9} + 9\ln|x^2| + c$$

Some Remarks on Strategy

Table of Formulas
pg. 495

Two Basic Techniques
- Substitution
- Integration by Parts

Several Useful Tricks
- Trig Formulas
- Trig Substitution
- Partial Fractions

Things to Try

- Simplify the Integrand
  - Multiply terms out $\int \sqrt{x}\,(1+x^2+5x)\,dx = \int\left[\sqrt{x}+x^{5/2}+5x^{3/2}\right]dx$
  - Rewrite trig formulas in terms of $\sin x, \cos x$

$$\int \frac{\cot\theta}{\csc^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta}\cdot\sin^2\theta\,d\theta$$

$$= \int \cos\theta\sin\theta\,d\theta$$

$$= \int \tfrac{1}{2}\sin 2\theta\,d\theta$$

$$= -\tfrac{1}{4}\cos 2\theta + C$$

- Look for a substitution

Find a function u in the integral whose derivative is also present

$$\int \frac{3x^2+2}{x^3+2x+7}\,dx$$

- Try denominators, terms raised to any power, inner functions, etc.

- Classify according to a type

① Trig functions

- Powers of $\sin x$ & $\cos x$
- Powers of $\tan x$ & $\sec x$
- Powers of $\csc x$ & $\cot x$

② Rational Functions

- Partial fractions

③ Product of a polynomial & exponential/transcendental

- Integration by parts

④ Radicals

① $\sqrt{\pm x^2 \pm a^2}$ → Trig substitution

② $\sqrt[n]{ax+b}$ → $u = \sqrt[n]{ax+b}$ Sometimes works for any root.

④ Fiddle

ⓐ Substitution (nonobvious)

ⓑ Parts on something other than a product

ⓒ Play with the integrand

$$\int \frac{dx}{1-\sin x} = \int \frac{dx}{1-\sin x}\left(\frac{1+\sin x}{1+\sin x}\right) = \int \frac{dx(1+\sin x)}{1-\sin^2 x}$$

$$= \int \frac{dx\,(1+\sin x)}{\cos^2 x}$$

$$= \int [\sec^2 x + \sec x \tan x]\,dx$$

$$= \tan x + \sec x + C$$

⑤ Multiple methods

Use parts, substitutions, etc. several times

Examples

$$\int e^{2\sqrt{x}}\,dx$$

$u = 2\sqrt{x}$

$du = \frac{1}{\sqrt{x}}\,dx$

$\frac{1}{2}u\,du = dx$

$$= \int e^u \tfrac{1}{2}u\,du$$

$$= \tfrac{1}{2}ue^u - \tfrac{1}{2}e^u + C$$

$$= \tfrac{1}{2}2\sqrt{x}e^{2\sqrt{x}} - \tfrac{1}{2}e^{2\sqrt{x}} + C$$

$\frac{1}{2}u$	e^u
$\frac{1}{2}$	e^u
0	e^u

(2) $\int \frac{x^5+1}{x^3-4x^2+3x}$ Partial Fractions

(3) $\int \frac{dx}{x^2\sqrt{\ln x}}$ $u=\ln x$

(4) $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+x}{1-x}}\left(\sqrt{\frac{1+x}{1+x}}\right)dx$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right)dx$$

$$= \sin^{-1}(1-x^2) - \sqrt{1-x^2} + C$$

$\frac{1}{\sqrt{x^2-1}}$	$\frac{2x}{\sqrt{x^2-1}}$	$\frac{x^2}{\sqrt{x^2-1}}$	$\frac{dx}{x^2\sqrt{x^2-1}}$	$\frac{2x}{x^2-1}$
$\sin^{-1}x$	substitute	partial fractions substitution	Trig substitution	Partial Fractions