Trigonometric Integration Lecture 4

We know how to integrate sin, cos, tan, and the derivatives of the trig functions.

$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Powers of sinx and cosx

$$\int \cos^3 x\,dx = \int \cos x\,(\cos^2 x)\,dx$$

- Substitution unhelpful, would need $\sin x\,dx$
- Use trig identities

$$= \int \cos x\,(1-\sin^2 x)\,dx$$

$$= \int [\cos x - \cos x \sin^2 x]\,dx$$

$$= \int \cos x\,dx - \int \cos x \sin^2 x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \sin x - \int u^2\,du$$

$$= \sin x - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

- So its useful to have terms of the form $\sin^k(x)\cos x$ or $\cos^k(x)\sin x$ (where k could be 0)
- It's enough to have an odd number of powers of one of them.

Example $\int \cos^5 x \sin^2 x\,dx = \int \cos x\,[\cos^2 x]^2 \sin^2 x\,dx$

$$= \int \cos x\,[1-\sin^2 x]^2 \sin^2 x\,dx$$

$$= \int \cos x\,[1 - 2\sin^2 x + \sin^4 x]\sin^2 x\,dx$$

$$= \int \cos x\,[\sin^2 x - 2\sin^4 x + \sin^6 x]\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int [u^2 - 2u^4 + u^6]\,du$$

$$= -\left(\tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7\right) + C$$

$$= -\left(\tfrac{1}{3}\cos^3 x - \tfrac{2}{5}\cos^5 x + \tfrac{1}{7}\cos^7 x\right) + C$$

What about examples with purely even powers? Use other trig identities.

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

(3) $$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi/2}$$ Note quick mental substitution

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\right]$$

$$= \frac{\pi - 1}{4}$$

(4) $$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

$$= \int \left[\tfrac{1}{2}(1+\cos(2x))\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos(2x) + \cos^2(2x)\right)dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\right)dx$$

$$= \frac{1}{4}\left[x + \sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x)\right] + C$$

$$= \frac{3}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Strategy–

$\int \sin^m x \cos^n x$

- If n is odd, turn all but one $\cos x$ into $\sin x$ using $\cos^2 x = 1 - \sin^2 x$
- Similarly if m is odd, turn all but one power of $\sin x$ into $\cos x$ using $\sin^2 x = 1 - \cos^2 x$
- If m, n even, use

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin x \cos x = \frac{1}{2}\sin 2x$$

We can use the other Pythagorean identities such as $\sec^2 x = 1 + \tan^2 x$.

(5) $\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x (\sec^2 x) \sec^2 x \, dx$

If we want $u = \tan x$, need $du = \sec^2 x \, dx$

$$= \int \tan^4 x (1 + \tan^2 x) \sec^2 x \, dx$$
$$= \int (u^4 + u^6) \, du$$
$$= \frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$
$$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

(6) $\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x (\sec x \tan x \, dx)$

If we want $u = \sec x$, need $du = \sec x \tan x \, dx$

$$= \int (1 - \sec^2 x)^2 \sec^2 x (\sec x \tan x \, dx)$$
$$= \int (1 - u^2)^2 u^2 \, du$$
$$= \int (1 - 2u^2 + u^4) u^2 \, du$$
$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sec^3 x - \frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + C$$

Strategy

$$\int \sec^m x \tan^n x$$

- If m is even, save a $\sec^2 x$ and convert the rest to $\tan x$ using $\sec^2 x = 1 + \tan^2 x$
- If n is odd, save a $\sec x \tan x$ and convert all other factors of $\tan x$ to $\sec x$ using $\tan^2 x = 1 - \sec^2 x$.
- Otherwise trickery

Recall $\int \tan x = -\ln|\cos x| + C = \ln|\sec x| + C$

Fact **Propn** (Very hard) $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Both $\sec x$ and $\tan x$ need to be multiplied by $\sec x$ to get to their derivatives.

$u = \sec x + \tan x$

$du = [\sec x \tan x + \sec^2 x] \, dx$

(7) $\int \tan^3 x\, dx = \int \tan x (\sec^2 x - 1)\, dx$

$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$

$= \frac{1}{2}\tan^2 x - \ln|\sec x| + c$

(8) $\int \sec^3 x = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$u = \sec x$ $\quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx$ $\quad v = \tan x$

$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + c$

$\int \sec^3 x\, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + c$

Notes: Powers are important because of volumes, polar coordinates

More Identities

$\int \sin(mx)\cos(nx)$, $\int \sin(mx)\sin(nx)$, $\int \cos(mx)\cos(nx)$

(a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

(d)

Example

$$\int \sin 3x \cos 7x \, dx = \int \left[\frac{1}{2}\left(\sin[(3-7)x] + \sin[(3+7)x]\right)\right] dx$$

$$= \frac{1}{2} \int [\sin(-4x) + \sin 10x] \, dx$$

$$= \frac{1}{2}\left(+\frac{1}{4} \cos x(-4x) - \frac{1}{10} \cos(10x) \right) + C$$

$$= \frac{1}{8} \cos(+4x) - \frac{1}{10} \cos(10x) + C$$

Trig Substitution: An application of the Pythagorean ~~identities~~

Example 1

① $\int \sqrt{a^2-x^2}\,dx = ?$

We know $\int x\sqrt{a^2-x^2}\,dx = \frac{-1}{2}\int \sqrt{u}\,du$

$u = a^2 - x^2$

$du = -2x\,dx$

$$= \frac{-1}{2}\left(\frac{2}{3}u^{3/2}\right) + c$$

$$= \frac{-1}{3}u^{3/2} + c$$

$$= \frac{-1}{3}(a^2-x^2) + c$$

Notice that x can only be between -1 and 1. What other functions are like this?

Trick: inverse substitution

We know

$$1 - \sin^2\theta = \cos^2\theta$$

$$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$

$x = a\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cancel{+}a\cos\theta\,d\theta$

$$\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\,(+a\cos\theta\,d\theta)$$

$$= \int\sqrt{a^2\cos^2\theta}\,(+a\cos\theta\,d\theta)$$

$$= \int(a\cos\theta)(+a\cos\theta\,d\theta)$$

Note $\cos\theta \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int +a^2\cos^2\theta\,d\theta$$

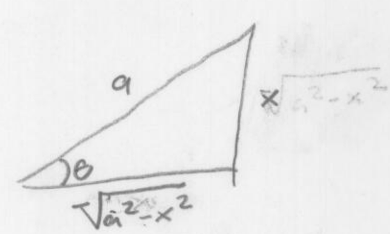
$$= \int +a^2\left(\tfrac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= \frac{+a^2}{2}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{+a^2}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + c$$

$$= \frac{+a^2}{2}\left(\theta + \underline{\sin\theta\cos\theta}\right) + c$$

$$= \frac{+a^2}{2}\left(\theta + \frac{\sqrt{a^2-x^2}}{a}\left(\frac{x}{a}\right)\right) + c$$



$$= \frac{1}{2}\left[-a^2 \cos^{-1}\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2}\right] + C$$

Pythagorean Identities

$\cos^2 x + \sin^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\cancel{\csc^2 x = 1 + \cot^2 x}$

Never use

Possible Substitutions

or vice versa

$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2 x = \cos^2 x$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1+\tan^2 x = \sec^2 x$
$\sqrt{x^2-a^2}$	$x = a\sec\theta \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2 x - 1 = \tan^2 x$

Example

② $$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$$

$x^2 + 9$

$\tan^2\theta + 1 = \sec^2\theta$

$9\tan^2\theta + 9 = 9\sec^2\theta$

$x = 3\tan\theta$

$x^2 = 9\tan^2\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sec\theta\,\cancel{\tan\theta\,d\theta}}{9\tan^{\cancel{2}}\theta\sqrt{9\tan^2\theta+9}}$$

$$= \int \frac{3\sec\theta\,d\theta}{3\tan\theta\sqrt{9\sec^2\theta}}$$

$$= \int \frac{\sec\theta\,d\theta}{3\tan\theta\,3|\sec\theta|}$$

$$= \int \frac{d\theta}{9\tan\theta}$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin\theta}\,d\theta$$

$u = \sin\theta$

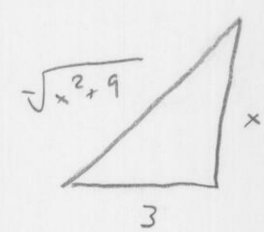
$du = \cos\theta$

$$= \frac{1}{9}\int \frac{1}{u}\,du$$

$$= \frac{1}{9}\ln|u| + c$$

$$= \frac{1}{9}\ln|\sin\theta| + c$$

$$= \frac{1}{9}\ln\left|\frac{x}{\sqrt{x^2+9}}\right| + C$$



$\tan\theta = \frac{x}{3}$

③ Not an example

$$\int \frac{x}{\sqrt{x^2+9}}\,dx$$ Should be done by substitution! (In general, look for a simple method first)

④ $$\int \frac{dx}{\sqrt{x^2-a^2}}, \quad a>0$$

$x = a\sec\theta \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$dx = a\sec\theta\tan\theta\, d\theta$

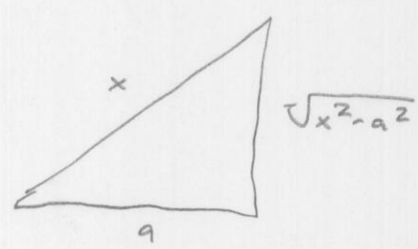
$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}}$$

$$= \int \frac{\cancel{a}\sec\theta\tan\theta\, d\theta}{\cancel{a}|\tan\theta|}$$ $\tan\theta > 0$ on the interval above

$\sec\theta = \frac{x}{a}$

$$= \int \sec\theta\, d\theta$$



$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c$$

Example

So far our substitution has been obvious. Let's try some harder cases.

(4)

$$\int \frac{x}{\sqrt{(12-4x-x^2)}}\,dx = \int \frac{x}{\sqrt{12+4-(x^2+4x+4)}}\,dx$$

Complete the square!

$$= \int \frac{x}{\sqrt{16-(x+2)^2}}\,dx$$

Let $u = x+2$

$du = dx$

$$= \int \frac{u-2}{\sqrt{16-u^2}}\,du$$

$u = 4\sin\theta$

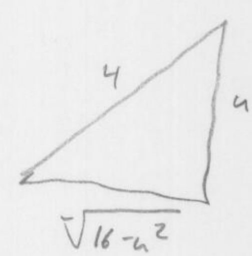
$du = 4\cos\theta\, d\theta$

$$= \int \frac{(4\sin\theta - 2)\,4\cos\theta\, d\theta}{\sqrt{16-16\sin^2\theta}}$$

$$= \int \frac{(4\sin\theta - 2)(4\cos\theta\, d\theta)}{\sqrt{16\cos^2\theta}}$$

$\frac{u}{4} = \sin\theta$

$$= \int \frac{(4\sin\theta - 2)(4\cos\theta\, d\theta)}{4\cos\theta}$$



$$= \int (4\sin\theta - 2)\,d\theta$$

$$= -4\cos\theta - 2\theta + c$$

$$= -4\left(\frac{\sqrt{16-u^2}}{4}\right) - 2\sin^{-1}\left(\frac{u}{4}\right) + C$$

$$= -4\left(\frac{\sqrt{16-(x+2)^2}}{4}\right) - 2\sin^{-1}\left(\frac{x+2}{4}\right) + C$$

(5)

$$\int_0^{\frac{2}{\sqrt{3}}} \frac{x^3 \, dx}{(9x^2+4)^{3/2}} = \int_0^{\frac{2}{\sqrt{3}}} \frac{x^3 \, dx}{\left(\sqrt{9x^2+4}\right)^3}$$

$u = 3x$

$du = 3x \, dx$

$$= \int_{0=u}^{\frac{2}{\sqrt{3}}=u} \frac{\left(\frac{u}{3}\right)^3}{\left(\sqrt{u^2+4}\right)^3} (3 \, du)$$

$u = 2\tan\theta$

$du = 2\sec^2\theta \, d\theta$

$$= \frac{1}{9} \int_{0=\theta}^{\frac{\pi}{3}} \frac{(2\tan\theta)^3}{\left(\sqrt{4\tan^2\theta+4}\right)^3} 2\sec^2\theta \, d\theta$$

$u=0 \rightsquigarrow \tan\theta = 0 \rightsquigarrow \theta = 0$

$u = 2\sqrt{3} \rightsquigarrow \tan\theta = \sqrt{3}$

$\rightsquigarrow \theta = \frac{\pi}{3}$

Numbers
slightly
changed
from
class
to
be
tidier

$$= \frac{16}{9} \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta \, d\theta}{\left(\sqrt{4\sec^2\theta}\right)^3}$$

$$= \frac{16}{9} \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\tan^3\theta \sec^2\theta}{(2|\sec\theta|)^3} \, d\theta$$

$$= \frac{2}{9} \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta} \, d\theta$$

$$= \frac{2}{9} \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^3\theta} (\cos\theta) \, d\theta$$

$$= \frac{2}{9} \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{(1-\cos^2\theta)}{\cos^2\theta} \sin\theta \, d\theta$$

$t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$$= \frac{2}{9} \int_{t=1}^{t=\frac{1}{2}} -\left(\frac{1-t^2}{t^2}\right) dt$$

$$= \frac{2}{9} \int_{t=\frac{1}{2}}^{t=1} (t^{-2} - 1) \, dt$$

$$= \frac{2}{9}\left[-t^{-1}-t\right]_{t=\frac{1}{2}}^{t=1}$$

$$= \frac{2}{9}\left[(-1-1)-(-2-1)\right]$$

$$= \frac{2}{9}\left[-2+3\right]$$

$$= \frac{2}{9}$$