Integration by Parts: Antidifferentiating the Product/Quotient Rules

$u\Delta v$	
u v	$v\Delta u$

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx}\left(f(x)g(x)\right) dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \underbrace{\int f(x)g'(x)\,dx}$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

or, more succintly

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$f(x) = u$

$g(x) = v$

How do we use this?

① $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + c$

$u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \sin x$

Notice that our goal is to make the integral <u>less</u> complicated, so we let u be the term that would get simpler if we differentiated.

$\int x\cos x\,dx = \frac{1}{2}x^2\sin x - \int \frac{1}{2}x^2\cos x\,dx$

$\uparrow$

Poor

$u = \sin x$ $\quad dv = x\,dx$

$du = \cos x\,dx$ $\quad v = \frac{1}{2}x^2$

② $\int \ln x\,dx = x\ln x - \int \frac{x}{x}\,dx$

$= x\ln x - x + c$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

Fact that we can't do this yet is a serious problem

How do we pick u? LIATE

Logarithmic
Inverse trig
Algebraic
Trigonometric
Exponential

Goes in order of how much complexity differentiating removes (with the constraint that dv be integrable)

Rapid Integration by Parts - A Useful Trick

Example

$$\int t^3 e^t dt = t^3e^t - \int 3t^2e^t dt$$

$$= t^3e^t - \left[3t^2e^t - \int 6te^t dt\right]$$

$$= t^3e^t - 3t^2e^t + \left[6te^t - \int 6e^t dt\right]$$

$$= t^3e^t - 3t^2e^t + 6te^t - \left[6e^t - \int 0\,dt\right]$$

$$= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C$$

t^3	e^t
$3t^2$	e^t
$6t$	e^t
6	e^t
0	e^t

For use when one term differentiates to zero and the other is infinitely integrable.

Example

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

x^2	$\sin x$
$2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

Cyclic Integration: A Further Useful Trick

Certain functions such as $\sin x$ & $\cos x$ have periodic derivatives

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x - \left[e^x(-\cos x) - \int e^x - \cos x \, dx \right]$$

$u = e^x \qquad dv = \cos x \, dx$

$du = e^x dx \qquad v = \sin x$

$u = e^x \qquad dv = \sin x \, dx$

$du = e^x dx \qquad v = -\cos x$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

Of course we can apply this to the definite integrals as well

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

Example

$$\int_0^1 \sin^{-1}x\,dx$$

$u = \sin^{-1}x \qquad dv = dx$

$du = \dfrac{1}{\sqrt{1-x^2}} \qquad v = x$

$$\int_0^1 \sin^{-1}x = x\sin^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2$

$du = -2x\,dx$

$$= \left(1\sin^{-1}(1) - 0\sin^{-1}(0)\right) - \int_{u=1}^{u=0} \frac{-\frac{1}{2}du}{\sqrt{u}}$$

$$= \frac{\pi}{2} + \frac{1}{2}\int_{u=1}^{u=0} u^{-\frac{1}{2}}$$

$$= \frac{\pi}{2} + \frac{1}{2}(2)u^{\frac{1}{2}}\Big|_{u=1}^{u=0}$$

$$= \frac{\pi}{2} + (0-1)$$

$$= \frac{\pi}{2} - 1$$

Reduction Formulas Can use integration by parts to prove useful general formulas concerning simplifying integrals

$$\int \sin^n x\,dx = \frac{-1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$u = \sin^{n-1}x \qquad dv = \sin x\,dx$

$du = (n-1)\sin^{n-2}x\cdot\cos x\,dx \qquad v = -\cos x$

Trigonometric Integration

We know how to integrate sin, cos, tan, and the derivatives of the trig functions.

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\sec x) = \sec x \tan x)$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\quad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Powers of sinx and cosx

$$\int \cos^3 x\,dx = \int \cos x(\cos^2 x)\,dx$$

- Substitution unhelpful, would need $\sin x\,dx$
- Use trig identities

$$= \int \cos x(1-\sin^2 x)\,dx$$

$$= \int [\cos x - \cos x \sin^2 x]\,dx$$

$$= \int \cos x\,dx - \int \cos x \sin^2 x\,dx$$

$u = \sin x$
$du = \cos x\,dx$

$$= \sin x - \int u^2\,du$$

$$= \sin x - \frac{1}{3}u^3 + c$$

$$= \sin x - \frac{1}{3}\sin^3 x + c$$

- So its useful to have terms of the form $\sin^k(x)\cos x$ or $\cos^k(x)\sin x$ (where k could be 0)
- It's enough to have an odd number of powers of one of them.

Example $\int \cos^5 x \sin^2 x\,dx = \int \cos x[\cos^2 x]^2 \sin^2 x\,dx$

$$= \int \cos x[1-\sin^2 x]^2 \sin^2 x\,dx$$

$$= \int \cos x[1 - 2\sin^2 x + \sin^4 x]\sin^2 x\,dx$$

$$= \int \cos x[\sin^2 x - 2\sin^4 x + \sin^6 x]\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int [u^2 - 2u^4 + u^6]\,du$$

$$= -\left(\tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7\right) + C$$ (7)

$$= -\left(\tfrac{1}{3}\cos^3 x - \tfrac{2}{5}\cos^5 x + \tfrac{1}{7}\cos^7 x\right) + C$$

What about examples with purely even powers? Use other trig identities.

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

(3) $\displaystyle\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\, dx$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi/2}$$ Note quick mental substitution

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\right]$$

$$= \frac{\pi - 1}{4}$$

(4) $\displaystyle\int \cos^4 x\, dx = \int (\cos^2 x)^2\, dx$

$$= \int \left[\tfrac{1}{2}(1+\cos(2x))\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos(2x) + \cos^2(2x)\right) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x))\right) dx$$

$$= \frac{1}{4}\left[x + \sin(2x) + \tfrac{1}{2}x + \tfrac{1}{8}\sin(4x)\right] + C$$

$$= \frac{3}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Strategy

$\int \sin^m x \cos^n x$

- If n is odd, turn all but one $\cos x$ into $\sin x$ using $\cos^2 x = 1 - \sin^2 x$
- Similarly if m is odd, turn all but one power of $\sin x$ into $\cos x$ using $\sin^2 x = 1 - \cos^2 x$
- If m, n even, use
$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$
$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

We can use the other Pythagorean identities such as $\sec^2 x = 1 + \tan^2 x$.

(5) $\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x (\sec^2 x) \sec^2 x \, dx$

If we want $u = \tan x$, need $du = \sec^2 x \, dx$

$$= \int \tan^4 x (1 + \tan^2 x) \sec^2 x \, dx$$
$$= \int (u^4 + u^6) \, du$$
$$= \tfrac{1}{5} u^5 + \tfrac{1}{7} u^7 + C$$
$$= \tfrac{1}{5} \tan^5 x + \tfrac{1}{7} \tan^7 x + C$$

(6) $\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x (\sec x \tan x \, dx)$

If we want $u = \sec x$, need $du = \sec x \tan x \, dx$

$$= \int (1 - \sec^2 x)^2 \sec^2 x (\sec x \tan x \, dx)$$
$$= \int (1 - u^2)^2 u^2 \, du$$
$$= \int (1 - 2u^2 + u^4) u^2 \, du$$
$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sec^3 x - \frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + C$$

Strategy

$$\int \sec^m x \tan^n x$$

- If m is even, save a $\sec^2 x$ and convert the rest to $\tan x$ using $\sec^2 x = 1 + \tan^2 x$
- If n is odd, save a $\sec x \tan x$ and convert all other factors of $\tan x$ to $\sec x$ using $\tan^2 x = 1 - \sec^2 x$.
- Otherwise trickery

Recall $\int \tan x = -\ln|\cos x| + C = \ln|\sec x| + C$

Propn $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

✓ $\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Both $\sec x$ and $\tan x$ need to be multiplied by $\sec x$ to get to their derivatives.

$u = \sec x + \tan x$

$du = [\sec x \tan x + \sec^2 x] \, dx$

⑦ $\int \tan^3 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$

$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$

$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C$

⑧ $\int \sec^3 x = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$u = \sec x \qquad dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx \qquad v = \tan x$

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

Notes: Powers are important because of volumes & polar coordinates

More Identities

$\int \sin(mx)\cos(nx)$, $\int \sin(mx)\sin(nx)$, $\int \cos(mx)\cos(nx)$

ⓐ $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

ⓑ $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

ⓒ $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$