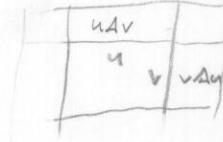


Integration by Parts: Antidifferentiating the Product/Quotient Rules

$$\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$



$$\int \frac{d}{dx} (f(x)g(x)) dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx$$

$$f(x)g(x) = \int f'(x)g(x) dx + \boxed{\int f(x)g'(x) dx}$$

$$\boxed{\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx}$$

Or, more succinctly

$$\boxed{\int u dv = uv - \int v du}$$

$$f(x) = u$$

$$g(x) = v$$

How do we use this?

$$\begin{aligned} ① \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

$$\begin{array}{ccc} u=x & dv=\cos x dx \\ du=dx & v=\sin x \end{array}$$

Notice that our goal is to make the integral less complicated, so we let u be the term that would get simpler if we differentiated.

$$\begin{aligned} \int x \cos x dx &= \frac{1}{2} x^2 \sin x - \int \frac{1}{2} x^2 \cos x dx \\ &\quad \uparrow \\ &\quad \text{Poor} \end{aligned}$$

$$\begin{array}{ccc} u=\sin x & dv=x dx \\ du=\cos x dx & v=\frac{1}{2} x^2 \end{array}$$

$$\textcircled{2} \quad \int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx$$

$$= x \ln x - x + C$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

Fact
that
we
can't do this
yet is a serious
problem

How do we pick u ? LIATE

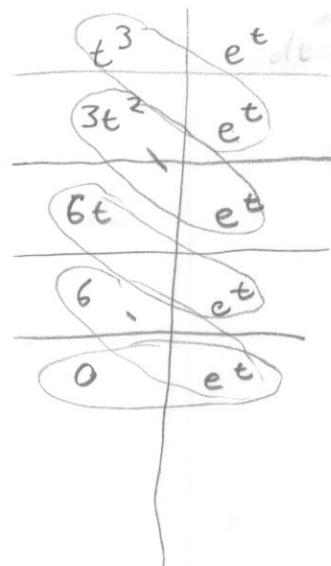
o	n	i	r	x
g	v	e	g	p
a	e	b	o	o
r	s	r	n	e
i	e	a	o	t
h	y	i	n	i
m	r	c	e	g
i	j	r	i	e
c	c	i	c	i

Goes in order
of how much
complexity
differentiating
removes
(with the constraint
that dv be integrable)

Rapid Integration by Parts - A Useful Trick

Example

$$\begin{aligned} \int t^3 e^t dt &= t^3 e^t - \int 3t^2 e^t dt \\ &= t^3 e^t - \left[3t^2 e^t - \int 6t e^t dt \right] \\ &= t^3 e^t - 3t^2 e^t + \left[6t e^t - \int 6e^t dt \right] \\ &= t^3 e^t - 3t^2 e^t + 6t e^t - \left[6e^t - \int 6dt \right] \\ &= t^3 e^t - 3t^2 e^t + 6t e^t - 6e^t + C \end{aligned}$$

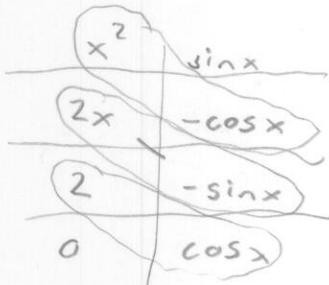


For use when one term differentiates to zero and the other is infinitely integrable.

Example

3

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Cyclic Integration: A Further Useful Trick

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \left[e^x (-\cos x) \right]$$

$$- \int e^x -\cos x dx$$

Certain functions such as $\sin x$ & $\cos x$ have periodic derivatives

$$u = e^x \quad dv = \cos x dx$$

$$du = e^x dx \quad v = \sin x$$

$$u = e^x \quad dv = \sin x dx$$

$$du = e^x dx \quad v = -\cos x dx$$

$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

of course we can apply this to the definite integrals as well

$$\int_a^b f(x)g'(x)dx = F(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx$$

Example

$$\int_0^1 \sin^{-1} x dx$$

$$u = \sin^{-1} x \quad dv = dx \\ du = \frac{1}{\sqrt{1-x^2}} \quad v = x$$

$$\int_0^1 \sin^{-1} x = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$u = 1-x^2 \\ du = -2x dx$$

$$= (1 \sin^{-1}(1) - 0 \sin^{-1}(0)) - \int_{u=1}^{u=0} \frac{-\frac{1}{2} du}{\sqrt{u}}$$

$$= \frac{\pi}{2} + \frac{1}{2} \int_{u=1}^{u=0} u^{-\frac{1}{2}} du$$

$$= \frac{\pi}{2} + \frac{1}{2} (2) u^{\frac{1}{2}} \Big|_{u=1}^{u=0}$$

$$= \frac{\pi}{2} + (0 - 1)$$

$$= \frac{\pi}{2} - 1$$

Reduction Formulas can use integration by parts to prove useful general formulas concerning simplifying integrals

$$\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$u = \sin^{n-1} x \quad dv = \sin x dx$$

$$du = (n-1) \sin^{n-2} x \cos x dx \quad v = -\cos x$$

Trigonometric Integration

We know how to integrate \sin , \cos , \tan , and the derivatives of the trig functions.

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Powers of $\sin x$ and $\cos x$

$$\int \cos^3 x dx = \int \cos x (\cos^2 x) dx$$

$$= \int \cos x (1 - \sin^2 x) dx$$

$$= \int [\cos x - \cos x \sin^2 x] dx$$

$$= \int \cos x dx - \int \cos x \sin^2 x dx \quad u = \sin x$$

$$= \sin x - \int u^2 du \quad du = \cos x dx$$

$$= \sin x - \frac{1}{3} u^3 + C$$

$$= \sin x - \frac{1}{3} \sin^3 x + C$$

- Substitution unhelpful, would need $\sin x dx$

- Use trig identities

- So it's useful to have terms of the form $\sin^k(x) \cos x$ or $\cos^k(x) \sin x$ (where k cannot be 0)
- It's enough to have an odd number of powers of one of them.

Example

$$\int \cos^5 x \sin^2 x dx = \int \cos x [\cos^2 x]^2 \sin^2 x dx$$

$$= \int \cos x [1 - \sin^2 x]^2 \sin^2 x dx$$

$$= \int \cos x [1 - 2\sin^2 x + \sin^4 x] \sin^2 x dx$$

$$= \int \cos x [\sin^2 x - 2\sin^4 x + \sin^6 x] dx$$

$$= - \int [u^2 - 2u^4 + u^6] du$$

$$u = \cos x \\ du = -\sin x dx$$

$$= -\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C$$

$$= -\left(\frac{1}{3}\cos^3x - \frac{2}{5}\cos^5x + \frac{1}{7}\cos^7x\right) + C$$

What about examples with purely even powers? Use other trig identities.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(3) $\int_0^{\pi/2} \sin^2 x dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) dx$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin(2x) \right]_0^{\pi/2} \quad \text{Note quick mental substitution}$$
$$= \frac{1}{2} \left[\left(\frac{\pi}{2} - \frac{1}{2} \sin\left(\frac{\pi}{2}\right)\right) - (0 - \frac{1}{2} \sin(0)) \right]$$
$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{1}{2} \right]$$
$$= \frac{\pi-1}{4}$$

(4) $\int \cos^4 x dx = \int (\cos^2 x)^2 dx$

$$= \int \left[\frac{1}{2}(1 + \cos(2x)) \right]^2 dx$$
$$= \frac{1}{4} \int (1 + 2\cos(2x) + \cos^2(2x)) dx$$
$$= \frac{1}{4} \int (1 + 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))) dx$$
$$= \frac{1}{4} \left[x + \sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x) \right] + C$$
$$= \frac{3}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Strategy

$$\int \sin^m x \cos^n x$$

- If n is odd, turn all but one $\cos x$ into $\sin x$ using $\cos^2 x = 1 - \sin^2 x$
- Similarly if m is odd, turn all but one power of $\sin x$ into $\cos x$ using $\sin^2 x \leftrightarrow 1 - \cos^2 x$
- If m, n even, use

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

We can use the other Pythagorean identities such as $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned}
 \textcircled{5} \quad \int \tan^4 x \sec^4 x dx &= \int \tan^4 x (\sec^2 x) \sec^2 x dx \\
 &= \int \tan^4 x (1 + \tan^2 x) \sec^2 x dx \\
 &= \int (u^4 + u^6) du \\
 &= \frac{1}{5} u^5 + \frac{1}{7} u^7 + C \\
 &= \frac{1}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C
 \end{aligned}$$

If we want
 $u = \tan x$,
need
 $du = \sec^2 x dx$

$$\begin{aligned}
 \textcircled{6} \quad \int \tan^5 x \sec^3 x dx &= \int \tan^4 x \sec^2 x (\sec x \tan x dx) \\
 &= \int (1 + \sec^2 x)^2 \sec^2 x (\sec x \tan x dx) \\
 &= \int (1 - u^2)^2 u^2 du \\
 &= \int (1 - 2u^2 + u^4) u^2 du \\
 &= \int (u^2 - 2u^4 + u^6) du
 \end{aligned}$$

If we want $u = \sec x$,
need $du = \sec x \tan x dx$

$$= \frac{1}{3} u^3 - \frac{2}{5} u^5 + \frac{1}{7} u^7 + C$$

$$= \frac{1}{3} \sec^3 x - \frac{2}{5} \sec^5 x + \frac{1}{7} \sec^7 x + C$$

Strategy

$$\int \sec^m x \tan^n x$$

- If m is even, save a $\sec^2 x$ and convert the rest to $\tan x$ using
 $\sec^2 x = 1 + \tan^2 x$

- If n is odd, save a $\sec x \tan x$ and convert all other factors of $\tan x$ to
 $\sec x$ using $\tan^2 x = 1 - \sec^2 x$.

* Otherwise trickery

Recall

$$\int \tan x = -\ln |\cos x| + C = \ln |\sec x| + C$$

$$\frac{\sin x}{|\cos x|}$$

Fact Propn (Very basic)

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\begin{aligned} \checkmark \int \sec x dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{du}{u} \\ &\stackrel{u = \sec x + \tan x}{=} \ln |u| + C \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

Both $\sec x$ and $\tan x$ need to be multiplied by $\sec x$ to get to their derivatives.

$$u = \sec x + \tan x$$

$$du = [\sec x \tan x + \sec^2 x] dx$$

$$\textcircled{7} \quad \int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \frac{1}{2} \tan^2 x - \ln |\sec x| + C$$

Example

$$\textcircled{8} \quad \int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$\begin{aligned} u &= \sec x & dv &\equiv \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x \end{aligned}$$

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$

Notes: Powers
are important
because of
volumes³, polar
coordinates

More Identities

$$\int \sin(mx) \cos(nx), \int \sin(mx) \sin(nx), \int \cos(mx) \cos(nx)$$

$$\textcircled{a} \quad \sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

$$\textcircled{b} \quad \sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\textcircled{c} \quad \cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$