Lecture 2

Last Time: Fundamental Theorem of Calculus

This time: Integrals as Net Change, Substitution

Net Change

We have $\int_a^b F(x)dx = F(b) - F(a)$

or $\int_a^b F'(x)dx = F(b) - F(a)$

Net change theorem

The integral of a rate of change is the net change

Examples

- $V(t)$ = volume of water in a lake at time t

  $V'(t)$ = rate at which water flows into (or out of) the lake

  So $\int_{t_1}^{t_2} V'(t) = V(t_2) - V(t_1)$ is the change in amount of water in the lake between time t_1 and time t_2

- If the rate of growth of a population is $\frac{dn}{dt}$, then

  $\int_{t_1}^{t_2} \frac{dn}{dt} dt = n(t_2) - n(t_1)$ is net change in the population from t_1 to t_2

Suppose $n(t) = .1(t) + .7\sin t)^{1}$ is pigeon population in thousands of pigeons in NYC where t is years after 1990. Then change in population from 2009 to 2011 is

$$= 21 - \frac{85}{2} + 30$$

$$= \frac{17}{2}$$

Distance Travelled

$$\int_1^4 |t^2 - 5t + 6|\,dt = \int_1^2 (t^2 - 5t + 6) + \int_2^3 -(t^2 - 5t + 6) + \int_3^4 (t^2 - 5t + 6)\,dt$$

$$|t^2 - 5t + 6| = \begin{cases} t^2 - 5t + 6 & 1 \le t \le 2 \\ -(t^2 - 5t + 6) & 2 \le t \le 3 \\ t^2 - 5t + 6 & 3 \le t \le 4 \end{cases}$$

$$= \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\right)\Big|_1^2 - \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\right)\Big|_2^3 + \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\right)\Big|_3^4$$

$$= \left[\left(\frac{8}{3} - 10 + 12\right) - \left(\frac{1}{3} - \frac{5}{2} + 6\right)\right]$$

$$- \left[\left(9 - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - 10 + 12\right)\right] + \left[\left(\frac{64}{3} - 40 + 24\right) - \left(9 - \frac{45}{2} + 18\right)\right]$$

$$= -36 + \frac{79}{3} + \frac{95}{2}$$

$$= \frac{217}{6}$$

$$= 36\frac{1}{6}$$

$$\int_{19}^{21} (.1t + .7\sin t)\,dt = .05t^2 + .7\cos t \Big|_{19}^{21}$$

Sidebar: Recall we <u>always</u> work in radians

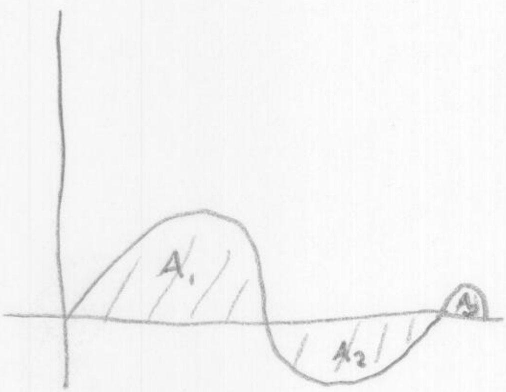
$$= .05(21^2 - 19^2) + .7(\cos 21 - \cos 19)$$

$$= 2.92$$

Increase of nearly 3000 pigeons.

- If an object moves in a straight line w/ position function $s(t)$, its velocity is $v(t) = s'(t)$ so

$$\int_{t_1}^{t_2} v(t)\,dt = \underbrace{s(t_2) - s(t_1)}_{\text{displacement}}$$



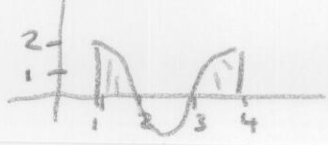
The distance an object actually travels is

$$\int_{t_1}^{t_2} |v(t)|\,dt$$

Distance $A_1 + A_2 + A_3$

Displacement $A_1 - A_2 + A_3$

<u>Example</u> A particle moving in a straight line has velocity $v(t) = t^2 - 5t + 6$
Find displacement & distance travelled on $1 \le t \le 4$



<u>Displacement</u> $\int_1^4 (t^2 - 5t + 6)\,dt = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_1^4$

$$= \left(\frac{64}{3} - 40 + 24\right) - \left(\frac{1}{3} - 5/2 + 6\right)$$

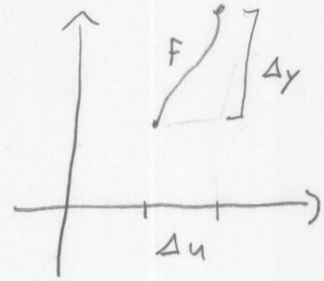
Substitution: Antidifferentiating the Chain Rule

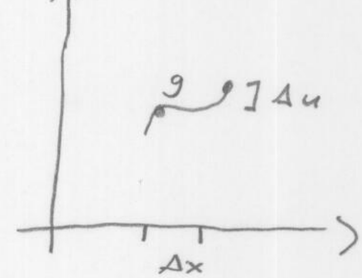
$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

or

$y = f(u)$ $\qquad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$u = g(x)$





Example 1

$y = e^{\sin x}$ $\qquad u = \sin x \qquad y = e^u$

$\dfrac{dy}{dx} = e^u \cdot \cos x$

$dy = e^{\sin x}(\cos x \, dx)$

So suppose we have

$$\int e^{\sin x} \cos x \, dx = \int e^u \, du$$

$u = \sin x$	$= e^u + c$
$du = \cos x \, dx$	$= e^{\sin x} + c$

In general suppose we have

$$\int f'(g(x)) \cdot g'(x) \, dx = f(g(x)) \qquad u = g(x)$$

$$du = g'(x) \, dx$$

$$\int F'(g(x))\,g'(x)\,dx = \int F'(u)\,du$$
$$= F(u) + c$$

Substitution Rule

If $u = g(x)$ is a differentiable function and f is cts on F,

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Examples:

(2) $$\int 3x^2\sqrt{1+x^3}\,dx = \int \sqrt{u}\,du$$

$u = 1+x^3$ $\qquad = \frac{2}{3}u^{3/2} + c$

$du = 3x^2$ $\qquad = \frac{2}{3}(1+x^3)^{3/2} + c$

(3) $$\int x\sin(3+x^2)\,dx = \int \sin u \cdot \left(\tfrac{1}{2}du\right)$$

Can only do this with a constant! Not x!

$u = 3+x^2$ $\qquad = -\frac{1}{2}\cos u + c$

$du = 2x\,dx$ $\qquad = -\frac{1}{2}\cos(3+x^2) + c$

$\frac{1}{2}du = x\,dx$

We make our integrals simpler by doing the antiderivative in multiple steps. Hardest part is finding the correct u.

'Look for a u whose differential is somewhere in the function

- Look for a u that looks like an inner function
- Try many things

More examples

9 $\int \sqrt{5x+4}\,dx = \int \sqrt{u} \cdot \frac{du}{5} = \frac{2}{3} u^{3/2} \cdot \frac{1}{5} + c$

Guess 1 : $u = 5x+4$

$du = 5dx$

$= \frac{2}{15}(5x+4)^{3/2} + c$

Guess 2

$u = \sqrt{5x+4}$

$du = \frac{5}{2\sqrt{5x+4}}\,dx$

$dx = \frac{2}{5}\sqrt{5x+4}\,du$

$= \frac{2}{5}u\,du$

$\int u \cdot \frac{2}{5} u\,du = \frac{2}{5}\int u^2\,du$

$= \frac{2}{15}u^3 + c$

$= \frac{2}{15}(5x+4)^{3/2} + c$

5 $\int \frac{x}{\sqrt{1-6x^2}}\,dx = \int \frac{-\frac{1}{12}du}{\sqrt{u}}$

$u = 1-6x^2$

$du = -12x\,dx$

$-\frac{1}{12}du = x\,dx$

$= \frac{-1}{12}\int u^{-1/2}\,du$

$= -\frac{1}{12}(2)u^{1/2} + c$

$= \frac{-1}{6}\sqrt{1-6x^2} + c$

$\int \frac{x}{\sqrt{1-x^2}} = (1-x^2)^{1/2} + c$ A subtlety

$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}x + c$

More exciting examples

6. Look for denominators

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int -\frac{du}{u}$$

$$= \ln|u| + c$$

$$= \ln|\cos x| + c$$

Sidebar: General antiderivative of $\frac{1}{x}$

7.

$$\int \sqrt{1+x^2} \cdot x^5 \, dx = \int \sqrt{1+x^2} \; x^4 \cdot x \, dx$$

we don't want to continue to have a complicated thing under a square root

$u = 1+x^2$

$du = 2x\,dx$

$x^2 = u-1$

$x^4 = u^2 - 2u + 1$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du$$

$$= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + c$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + c$$

Definite Integrals

If we're ultimately looking for a number, we never need to go back to x.

$$\int_0^1 \sqrt{5x+4}\,dx = \int_{u=4}^{u=9} \sqrt{u} \cdot \frac{1}{5}\,du$$

$u = 5x+4$

$du = 5\,dx$

$$= \frac{2}{15} u^{3/2} \Big|_4^9$$

$$= \frac{2}{15}(27-8)$$

$$= \frac{38}{15}$$

Substitution Rule For Definite Integrals

If g' is continuous on $[a,b]$, and f is cts on the range of $u=g(x)$, then

$$\int_a^b F(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof If F is an antiderivative of f, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, so by FTC part II

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

$$\wr$$

$$\int_a^b f(u)\,du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Examples

① $$\int_1^2 \frac{dx}{(7-2x)^2} = \int_{u=5}^{u=3} \frac{-\frac{1}{2}\,du}{u^2}$$

$u = 7-2x$

$du = -2dx$

$$= \int_{u=5}^{u=3} \frac{-1}{2}u^{-2}\,du$$

$$= \frac{1}{2}u^{-1}\Big|_5^3$$

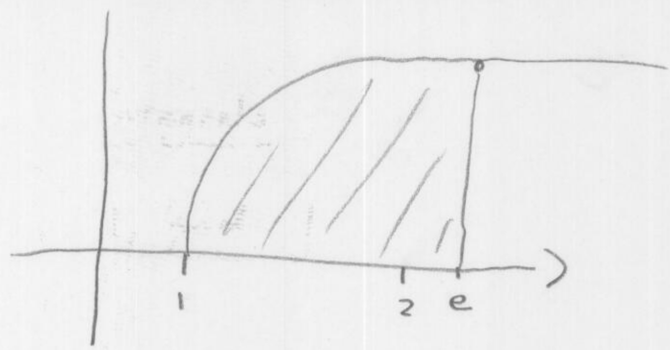
$$= \frac{1}{2}\left(\frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{5}\right)$$

$$= \frac{1}{15}$$

② $\int_1^e \frac{\ln x}{x}\,dx = \int_{u=0}^{u=1} u\,du$

$u = \ln x$ $\quad = \frac{1}{2}u^2 \Big|_0^1$

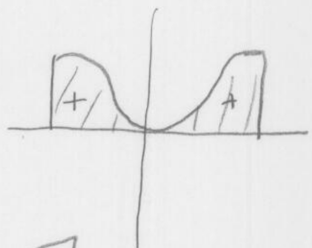
$du = \frac{1}{x}\,dx$ $\quad = \frac{1}{2}$



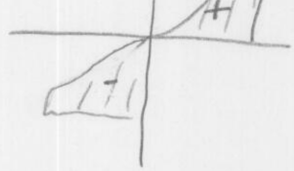
Integrals & Symmetry

Let f be continuous on $[-a, a]$.

Thm ① If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



② If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$



Proof

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

Let $u = -x$
$du = -dx$

$$= -\int_{0}^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_{u=0}^{u=a} -f(-u)\,du + \int_0^a f(x)\,dx$$

$$= \int_0^a f(-u)\,du + \int_0^a f(u)\,du$$

If f even $f(u) = f(-u)$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

If f odd $f(-u) = -f(u)$

$$\int_{-a}^{a} f(x)\,dx = \underbrace{-\int_0^a f(u)\,du + \int_0^a f(x)\,dx}_{=0}$$

Examples

① $\int_{-1}^{1} (x^8+1)dx = 2\int_0^1 (x^8+1)dx$

$= 2\left[\frac{1}{9}x^9 + x\right]_0^1$

$= 2\left(\frac{1}{9}+1\right)$

$= \frac{20}{9}$

$f(x) = x^8+1$

$f(-x) = (-x)^8+1$

$= x^8+1$

$= f(x)$

② $\int_{-\pi}^{\pi} \frac{\tan x}{1+x^4+x^6}\,dx = 0$

$F(x) = \frac{\tan x}{1+x^4+x^6}$

$F(-x) = \frac{\tan(-x)}{1+(-x)^4+(-x^6)}$

$= \frac{-\tan x}{1+x^4+x^6}$