Lecture 2

Last Time: Fundamental Theorem of Calculus
This time: Integrals as Net Change, Substitution

**Net Change**

We have \[ \int_a^b f(x) \, dx = F(b) - F(a) \]

or \[ \int_a^b F'(x) \, dx = F(b) - F(a) \]

**Net change theorem**

The integral of a rate of change is the net change.

**Examples**

- \( V(t) \) = volume of water in a lake at time \( t \)
  
  \( V'(t) \) = rate at which water flows into (or out of) the lake

  So \[ \int_{t_1}^{t_2} V'(t) \, dt = V(t_2) - V(t_1) \] is the change in amount of water in the lake between time \( t_1 \) and time \( t_2 \).

- If the rate of growth of a population is \( \frac{dn}{dt} \), then

  \[ \int_{t_1}^{t_2} \frac{dn}{dt} \, dt = n(t_2) - n(t_1) \] is net change in the population from \( t_1 \) to \( t_2 \).

  Suppose \( n(t) = e^{(t + 0.75 \sin t)} \) where \( t \) is years after 1990.

  Then change in population from 2009 to 2011 is
Distance Travelled

\[ S_1 = \int_1^4 \left( t^2 - 5t + 6 \right) dt = \int_1^2 \left( t^2 - 5t + 6 \right) dt + \int_2^3 \left( t^2 - 5t + 6 \right) dt + \int_3^4 \left( t^2 - 5t + 6 \right) dt \]

\[ \left. \begin{align*}
&= \left[ \frac{1}{3} t^3 - \frac{5}{2} t^2 + 6t \right]_1^2 - \left[ \frac{1}{3} t^3 - \frac{5}{2} t^2 + 6t \right]_2^3 + \left[ \frac{1}{3} t^3 - \frac{5}{2} t^2 + 6t \right]_3^4 \\
&= \left[ \left( \frac{8}{3} - 10 + 12 \right) - \left( \frac{1}{3} - \frac{5}{2} + 6 \right) \right] \\
&\quad + \left[ \left( 4 - \frac{45}{2} + 18 \right) - \left( \frac{8}{3} - 10 + 12 \right) \right] \\
&\quad + \left[ \left( \frac{64}{3} - 40 + 24 \right) - \left( 4 - \frac{45}{2} + 18 \right) \right] \\
&= -36 + \frac{71}{3} + \frac{95}{2} \\
&= \frac{217}{3} - \frac{95}{2} \\
&= \frac{361}{6} \\
&= \frac{361}{6}
\]
\[ \int_{19}^{21} \left( 1.1t + 7 \sin t \right) dt = 0.05t^2 + 0.7 \cos t \bigg|_{19}^{21} \]
\[ = 0.05(21^2 - 19^2) + 0.7(\cos 21 - \cos 19) \]
\[ = 2.92 \]

Increase of nearly 3000 pigeons.

'If an object moves in a straight line with position function \( s(t) \), its velocity is \( v(t) = s'(t) \) so
\[ \int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1) \]

\[ \text{displacement} \]

The distance an object actually travels is
\[ \int_{t_1}^{t_2} |v(t)| \, dt \]

Distance \( A_1 + A_2 + A_3 \)
Displacement \( A_1 - A_2 + A_3 \)

**Example** A particle moving in a straight line has velocity \( v(t) = t^2 - 5t + 6 \)
Find displacement & distance travelled on \( 1 \leq t \leq 4 \)

\[ \text{Displacement} \quad \int_{1}^{4} (t^2 - 5t + 6) \, dt = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t \bigg|_{1}^{4} \]
\[ = \left( \frac{64}{3} - 40 + 24 \right) - \left( \frac{1}{3} - \frac{5}{2} + 6 \right) \]
Substitution: Antidifferentiating the Chain Rule

\[ \frac{d}{dx} (F(g(x))) = F'(g(x)) \cdot g'(x) \]

or

\[ y = f(u) \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Example 1

\[ y = e^{\sin x} \quad u = \sin x \quad y = e^u \]

\[ \frac{dy}{dx} = e^u \cdot \cos x \]

\[ dy = e^{\sin x} (\cos x \, dx) \]

So suppose we have

\[ \int e^{\sin x} \cos x \, dx = \int e^u \, du \]

\[ u = \sin x \quad = e^u + C \]

\[ du = \cos x \, dx \quad = e^{\sin x} + C \]

In general suppose we have

\[ \int f'(g(x)) \cdot g'(x) \, dx = F(g(x)) \quad u = g(x) \]

\[ du = g'(x) \, dx \]
\[ \int f'(g(x))g'(x) \, dx = \int f'(u) \, du = f(u) + C \]

Substitution Rule

If \( u = g(x) \) is a differentiable function and \( f \) is cts on \( F \),

\[ \int f(g(x))g'(x) \, dx = \int f(u) \, du \]

Examples:

2. \[ \int 3x^2 \sqrt{1+x^3} \, dx = \int \sqrt{u} \, du \]

\[ u = 1 + x^3 \]
\[ du = 3x^2 \, dx \]
\[ = \frac{2}{3} u^{3/2} + C \]
\[ = \frac{2}{3} (1+x^3)^{3/2} + C \]

3. \[ \int x \sin(3+x^2) \, dx = \int \sin u \cdot (\frac{1}{2}du) \]

\[ u = 3 + x^2 \]
\[ du = 2x \, dx \]
\[ \frac{1}{2} du = x \, dx \]

Can only do this with a constant! Not \( x \)!

We make our integrals simpler by doing the antiderivative in multiple steps. Hardest part is finding the correct \( u \).

'Look for a \( u \) whose differential is somewhere in the function'
Try many things

More examples

4 \[ \int \sqrt{5x+4} \, dx = \int \sqrt{u} \cdot \frac{du}{5} = \frac{2}{3} u^{3/2} \cdot \frac{1}{5} + C \]

Guess 1: \( u = 5x+4 \)
\[ du = 5 \, dx \]
\[ \frac{2}{15} (5x+4)^{3/2} + C \]

Guess 2: \( u = \sqrt{5x+4} \)
\[ du = \frac{5}{2 \sqrt{5x+4}} \, dx \]
\[ dx = \frac{2}{5} \sqrt{5x+4} \, du \]
\[ \frac{2}{15} u^{3/2} + C \]

\[ \int \frac{x}{\sqrt{1-6x^2}} \, dx = \int \frac{-1}{12} \frac{du}{\sqrt{u}} \]

\( u = 1 - 6x^2 \)
\[ du = -12x \, dx \]
\[ \frac{-1}{12} \frac{du}{x} = -\frac{1}{12} (2) u^{1/2} + C \]
\[ = \frac{-1}{6} \sqrt{1-6x^2} + C \]

\[ \int \frac{x}{\sqrt{1-x^2}} = (1-x^2)^{1/2} + C \]

A Subtlety

\[ \int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C \]
More exciting examples

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \]

\[ u = \cos x \]
\[ du = -\sin x \, dx \]
\[ = \int -\frac{du}{u} \]
\[ = \ln |u| + c \]
\[ = \ln |\cos x| + c \]  

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Sidebar: General antiderivative of \( \frac{1}{x} \)

\[ \int \sqrt{1+x^2} \, dx = \int \sqrt{1+x^2} \cdot x \, dx \]

\[ u = 1+x^2 \]
\[ du = 2x \, dx \]
\[ x^2 = u - 1 \]
\[ x = \sqrt{u - 1} \]

\[ = \frac{1}{2} \int \sqrt{u} (u - 2u + 1) \, du \]
\[ = \frac{1}{2} \left( \frac{2}{7} u^{3/2} - \frac{4}{5} u^{1/2} + \frac{2}{3} u^{3/2} \right) + c \]
\[ = \frac{1}{7} (1+x^2)^{3/2} - \frac{2}{3} (1+x^2)^{1/2} + \frac{1}{3} (1+x^2)^{3/2} + c \]

Definite Integrals

If we're ultimately looking for a number, we never need to go back to \( x \).

\[ \int_0^1 \sqrt{5x+4} \, dx = \int_{u=4}^{u=9} \sqrt{u} \cdot \frac{1}{5} \, du \]
\[ = \frac{2}{15} (27 - 8) \]
\[ = \frac{38}{15} \]
Substitution Rule for Definite Integrals

If \( g' \) is continuous on \([a, b]\), and \( f \) is cts on the range of \( u = g(x) \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

Proof. If \( F \) is an antiderivative of \( f \), \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \), so by FTC part II

\[
\int_a^b f(g(x))g'(x) \, dx = F(g(x)) \bigg|_a^b = F(g(b)) - F(g(a))
\]

\[
\int_a^b f(u) \, du = F(u) \bigg|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))
\]

Examples

1. \[
\int_1^2 \frac{dx}{(7 - 2x)^2} = \int_1^3 \frac{-\frac{1}{2} \, du}{u^2}
\]
   
   \( u = 7 - 2x \)
   \[ du = -2 \, dx \]
   
   \[
   = \int_{u=5}^{u=3} \frac{-\frac{1}{2} \, du}{u^2} = \frac{1}{2} u^{-1} \bigg|_5^3 = \frac{1}{2} (\frac{1}{3}) - \frac{1}{2} (\frac{1}{5}) = \frac{1}{15}
   \]
\[ 1. \quad \int_1^e \frac{\ln x}{x} \, dx = \int_{u=0}^{u=1} u \, du \]

\[ u = \ln x \]
\[ du = \frac{1}{x} \, dx \]
\[ = \frac{1}{2} \left[ 1 \right]_1^2 \]

\[ \text{Integrals Symmetry} \]

Let \( F \) be continuous on \([-a, a]\).

\text{Thm} \quad 1. \quad \text{IF } F \text{ is even, } \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx.

2. \quad \text{IF } f \text{ is odd, } \int_{-a}^{a} f(x) \, dx = 0.

\text{Proof}

\[ \int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

\[ = -\int_{0}^{a} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

\[ = -\int_{u=0}^{u=a} f(u) \, du + \int_{0}^{a} f(x) \, dx \]

\[ = \int_{0}^{a} f(u) \, du + \int_{0}^{a} f(u) \, du \]

If \( f \) is even, \( f(u) = f(-u) \)

\[ \Rightarrow \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \]

\[ \text{IF } f \text{ odd, } f(-u) = -f(u) \]

\[ \int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

\[ = -\int_{0}^{a} f(x) \, dx + \int_{0}^{a} f(x) \, dx \]

\[ = \int_{0}^{a} f(u) \, du + \int_{0}^{a} f(u) \, du \]
\[
\int \frac{\tan x}{1 + x^2} dx = \frac{1}{2} \ln|\tan x + x| + C
\]

\[
\int \frac{\sec^2 x}{1 + x^2} dx = \frac{1}{2} \ln|\sec^2 x + x| + C
\]