How can we produce new power series models from old?

We could integrate & differentiate...

Thm (Term by Term Integration & Differentiation)

If the power series $\sum c_n (x-a)^n$ has radius of convergence $R>0$, then

$$F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

is differentiable (ergo cts) on $(a-R, a+R)$ and

① $$F'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + 4c_4 (x-a)^3 + \cdots$$
$$= \sum_{n=1}^{\infty} n c_n (x-a)^n$$

② $$\int F(x)\,dx = C + c_0 (x-a) + \frac{c_1}{2} (x-a)^2 + \frac{c_2}{3} (x-a)^3 + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Radii of convergence are R for both power series.

That is

- $$\frac{d}{dx}\left[\sum c_n (x-a)^n\right] = \sum c_n \frac{d}{dx} (x-a)^n$$

- $$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx$$

Note: • Only holds for power series (not functions defined by series in general).

• Interval of convergence (i.e. endpoint behavior) might change.

Examples

1. Bessel Function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2}$$

2. Write $\frac{1}{(1-x)^2}$ as a power series.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{d}{dx}\left(1 + x + x^2 + \cdots\right)$$

$$= 1 + 2x + 3x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)x^n$$

Radius of convergence is still 1.

Examples

① Power series for $\ln(1+x)$?

$$\ln(1+x) = \int \frac{dx}{1+x}$$

$$= \int \frac{dx}{1-(-x)}$$

$$= \int [1 - x + x^2 - x^3 + \dots]\, dx \qquad |x| < 1$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

Remember when we claimed alternating harmonic series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln(2)$? This is why

②

What about $\ln(x)$?

$$\ln x = \ln((x-1)+1)$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-1)^n$$

Converges on $|x-1| < 1 \rightsquigarrow (0,2)$.

Example Power Series Representation for $f(x) = \tan^{-1} x$

$$\tan^{-1} x = \int \frac{dx}{1+x^2}$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{on } |x| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Example Find $\int_0^{.2} \frac{1}{1+x^6}\, dx$ to six decimal places.

$$\frac{1}{1+x^6} = \frac{1}{1-(-x^6)}$$

$$= \sum_{n=0}^{\infty} (-x^6)^n \qquad \text{on } |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{6n}$$

$$\int_0^{.2} \frac{dx}{1+x^6} = \int_0^{.2} \sum_{n=0}^{\infty} (-1)^n x^{6n}$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{6n+1} \Bigg]_0^{.2}$$

$$= \left[C + x - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \dots \right]_0^{.2}$$

$$= .2 - \frac{(.2)^7}{7} + \frac{(.2)^{13}}{13} - \frac{(.2)^{19}}{19} + \dots$$

Alternating

So $R_n < b_{n+1}$.

Want

$$S = s_n + R_n$$

$$R_n < 10^{-7}$$

Know

$$R_n < b_{n+1} = \frac{(.2)^{6(n+1)+1}}{6(n+1)+1} < 10^{-7}$$

$$(6(n+1)+1)\, 5^{6(n+1)+1} > 10{,}000{,}000$$

Satisfied by $n = 1$

$$\int_0^{.2} \frac{dx}{1+x^6} = .2 - \frac{(.2)^7}{7}$$

$$= .199998$$

Taylor and Maclaurin Series: A More Algorithmic Representation of Ftns by Power Series

So far: Figured out some functions by building up from others

Next: More systematic study

Assume f can be represented by a power series near a,

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \qquad |x-a| < R$$

What do coefficients need to be? Use the fact that computing at $x=a$ is easy

$$f(a) = c_0 + 0 + 0 + 0 + \cdots$$

$$\boxed{f(a) = c_0}$$

what happens if we differentiate?

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \qquad |x-a| < R$$

$$\boxed{f'(a) = c_1}$$

<u>Repeat</u>

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots \qquad |x-a| < R$$

$$f''(a) = 2c_2$$

$$\boxed{\frac{f''(a)}{2} = c_2}$$

$$f'''(x) = 6c_3 + 24c_4(x-a) + 60c_5(x-a)^2 + \cdots \qquad |x-a| < R$$

$$f'''(a) = 6c_3$$

$$\boxed{\frac{f'''(a)}{3!} = c_3}$$

In general $f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n\, c_n = n!\, c_n$

$$\boxed{\frac{f^{(n)}(a)}{n!} = c_n}$$

<u>Thm</u> If f has a power series expansion at a,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a_n)^n \qquad |x-a| < R,$$

then $c_n = \dfrac{f^{(n)}(a)}{n!}$

So $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

A power series in this form is a <u>Taylor series</u>.

(After Brook Taylor, 1715 work on series) Known to Gregory, Bernoulli

If $a=0$, it is a <u>Maclaurin series</u>

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= F(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

(After Colin Maclaurin, who popularized their use)

<u>Warning</u> <u>If</u> f has a power series representation, then it is its Taylor series... but there are functions which don't have a power series representation & aren't.

<u>Eg</u> $f(x) = \begin{cases} 0 & x<0 \\ e^{-x^2} & x>0 \end{cases}$ about 0

<u>Eg</u> Any function not infinitely differentiable

<u>Examples</u> Find the Maclaurin series

① $f(x) = e^x$

$f^{(n)}(x) = e^x$

$f^{(n)}(0) = 1$

<u>So</u> $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Radius of convergence?

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{n+1} \right|$$

$$= 0$$

Converges on $(-\infty, \infty)$

So if e^x has a power series expansion, this is it. Does it?

To wit, when is a function equal to the sum of its Taylor series?

Partial sums are Taylor polynomials

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In this case

$T_1(x) = 1 + x$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

So we're asking if $\lim_{n \to \infty} T_n(x) = f(x)$.

Equivalently $R_n(x) = f(x) = T_n(x)$

question is whether $\lim_{n \to \infty} R_n(x) = 0$

<u>Thm</u> If $F(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomials and $\lim_{n \to \infty} R_n(x) = 0$ for $|x-a| < R$, then $f(x)$ is equal to the sum of its Taylor series on $|x-a| < R$

How would we ever show this?

<u>Taylor's Inequality</u>

If $|f^{n+1}(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

<u>Idea of Proof</u> (when $n=1$)

Assume $|f''(x)| \leq M$ on $|x-a| \leq d$. Logical thing to

So for $a \leq x \leq a+d$,

$$\int_a^x f''(t)\,dt \leq \int_a^x M\,dt$$

$$f'(x)\Big|_a^x \leq M(x-a)$$

$$f'(x) - f'(a) \leq M(x-a)$$

$$f'(x) \leq f'(a) + M(x-a)$$

<u>Again!</u>

$$\int_a^x f'(t)\,dt \leq \int_x^a [f'(a) + M(x-a)]\,dx$$

$$f(x) - f(a) \leq f'(a)(x-a) + \frac{M}{2}(x-a)^2$$

$$f(x) \underbrace{- f(a) - f'(a)(x-a)} \leq \frac{M}{2}(x-a)^2$$

$$f(x) - (T_1(x)) \leq \frac{M}{2}(x-a)^2$$

$$R_1(x) \leq \frac{M}{2}(x-a)^2$$

<u>Similarly</u> $f''(x) \geq -M \rightsquigarrow R_1(x) \geq -\frac{M}{2}(x-a)^2$

so $|R_1(x)| \leq \frac{M}{2}|x-a|^2$

$\square$

Case $n \geq 1$: Many more integrals.

<u>Examples</u> Prove $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$f(x) = e^x$

$f^{(n+1)} = e^x$

For $d > 0$, if $|x| \leq d$, $|f^{(n+1)}(x)| = e^x \leq e^d = M$.

$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}$ for $|x| \leq e^d$

$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$

$\lim_{n \to \infty} \frac{x^n}{n!} = 0$

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^d \lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \checkmark$$

<u>In particular</u>

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots$$