Lecture 18

What do we do if we meet a series in the wild? We have a lot of tests available.

Classify according to form

1. $\sum \frac{1}{n^p}$ – convergent if $p > 1$
   – divergent if $p < 1$

2. $\sum ar^{n-1}$ or $\sum ar^n$ – geometric

3. Rational or algebraic function of n –
   - Compare to a p-series of geometric series
   - If necessary, apply comparison tests to $\sum |a_n|$

4. If $\lim_{n\to\infty} a_n \neq 0$, diverges

5. Alternating series $\sum (-1)^n b_n$ or $\sum (-1)^{n-1} b_n$ –

   Alternating series test

6. Involves factorials or other products (such as 3^n)

   – Ratio test

   – Doesn't work for rational & algebraic functions of n, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \to 1$

7. $\sum (b_n)^n$

   – Root test

8. $a_n = f(n)$ for $\int_1^\infty f(x)\,dx$ possible to evaluate

- f cts, positive, decreasing $\rightsquigarrow$ Integral test

Examples

1. $\sum_{n=1}^{\infty} \frac{n-2}{3n-4}$ $\quad \lim_{n\to\infty} \frac{n-2}{3n-4} = \frac{1}{3} \neq 0$ Test For Divergence

2. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{3n^3+5n^2+2}$ $\quad$ Algebraic Function $\rightsquigarrow$ (Limit) Comparison to a p-series $\sum \frac{n^2}{3n^3} = \sum \frac{1}{3n}$

3. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

   Can integrate $\int_1^\infty x^2 e^{-x^2}$ $\rightsquigarrow$ Integral Test. (Possibly also Ratio)

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+1}{2n^4+5}$ $\quad$ Alternating series $\rightsquigarrow$ Alternating Series Test

5. $\sum_{k=1}^{\infty} \frac{3^k}{k!}$ $\quad$ Factorials, powers $\rightsquigarrow$ Ratio Test

6. $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ $\quad$ Compare w/ $\sum_{n=1}^{\infty} \frac{1}{3^n}$

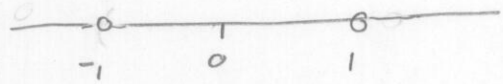
Intro to Power Series

<u>Goal</u> Approximate arbitrary cts functions by polynomials.

Motivation

$$\sum_{n=0}^{\infty} x^n = 1 + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

So on



, $f(x) = \frac{1}{1-x}$ is equal

to $\sum_{n=0}^{\infty} x^n$. We can approximate the function itself

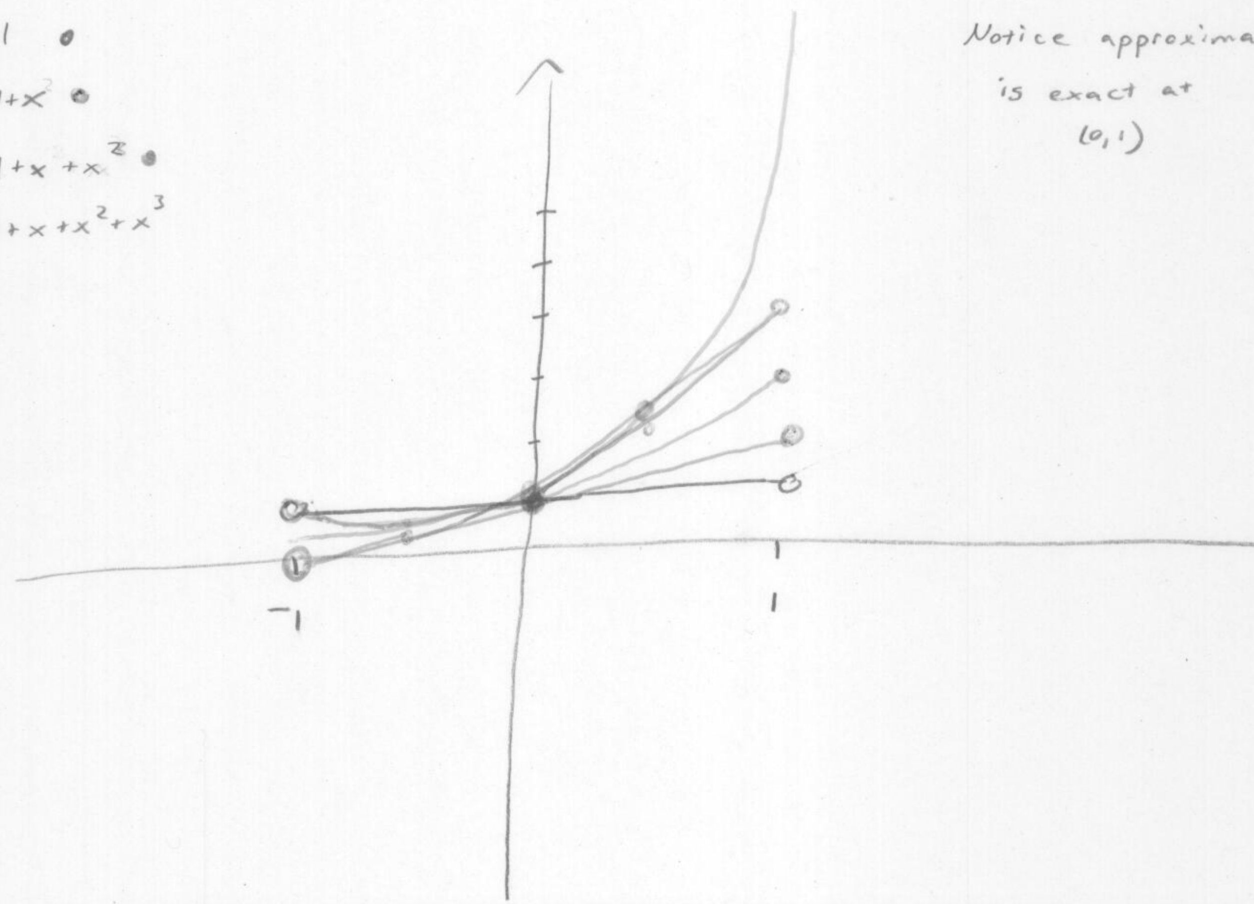
by partial sums of the series.

$f_1(x) = 1$

$f_2(x) = 1+x$

$f_3(x) = 1+x+x^2$

$f_3(x) = 1+x+x^2+x^3$



Notice approximation is exact at $(0,1)$

What if we could do this for a trig function, or a logarithm? That would be excellent, especially if we could control the error.

Intro to Power Series

Goal: Approximates arbitrary cts fcns by polynomials.

<u>Defn</u> A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$c_1, \ldots, c_n \in \mathbb{R}$

x a variable

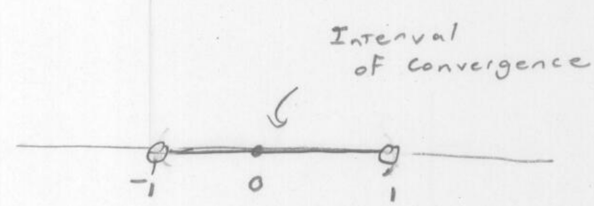
- Can test for convergence or divergence at each particular x
- Infinite analog of a polynomial.
- Might be convergent for some x & not others

<u>Example</u>

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{Geometric}$$

- Diverges when $|x| \geq 1$

Converges to $\dfrac{1}{1-x}$ when $|x| < 1$



<u>More general form</u>

<u>Defn</u> $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

is a <u>power series centered at a</u>, or <u>power series about a</u>.

When <u>$x=a$</u> the series is $0+0+0+0+\cdots = 0$.

Examples For what values of x does the series converge?

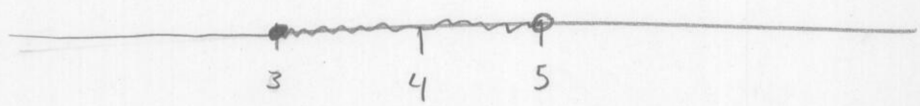
1. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{n}$

Ratio Test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{n+1}\cdot\frac{n}{(x-4)^n}\right|$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)(x-4)$$

$$= |x-4|$$

So absolutely convergent when $|x-4|<1$, divergent when $|x-4|>1$.



$x=3?$ Have to test independently.

$\displaystyle\sum_{n=1}^{\infty}\frac{(3-4)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges

$x=5?$

$\displaystyle\sum_{n=1}^{\infty}\frac{(5-4)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges

2. $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$

Ratio Test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$ for all $x\neq 0$

Diverges everywhere but 0.

Example (The Bessel Function) We can treat a power series as a function of x

Detects oscillation (e.g. of planetary bodies, drums, etc.)

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}\right|$$

$$= \left|\frac{x^2}{2^2 \cdot (n+1)^2}\right|$$

$$= \left(\frac{x}{4(n+1)}\right)^2$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{x}{4(n+1)}\right)^2 = 0 \quad \text{Convergent for all } x$$

This function (which is an infinite polynomial) can be approximated by its finite sums. See lovely picture pg. 743

Thm For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities.

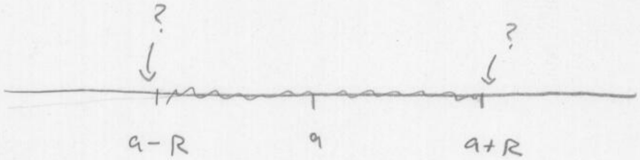
(i) Series converges only at $x = a$

(ii) Series converges on $(-\infty, \infty)$

(iii) There is an $R > 0$ such that the series converges if $|x-a| < R$ & diverges if $|x-a| > R$

- R is the <u>radius of convergence</u>

- We say the <u>interval of convergence</u> is the set of values for which the series converges



Endpoints have to be decided individually

<u>Could be</u>

$(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R)$ $[a-R, a+R]$

<u>Strategy</u>

- Use Ratio/Root Test to get R. Fails on endpoints.
- Check endpoints directly.

<u>Examples</u>

Find radius & invervals of convergence.

① $\sum_{n=0}^{\infty} \frac{(-4)^n x^n}{\sqrt{n+1}}$

<u>Ratio Test</u>

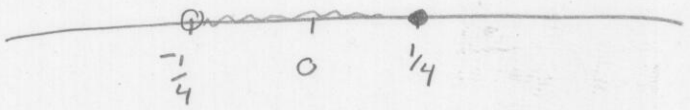
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-4)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-4)^n x^n}\right|$$

$$= \lim_{n\to\infty} \left| -4x \cdot \sqrt{\frac{n+1}{n+2}} \right|$$

Algebraic functions don't matter

$$= 4|x|$$

Converges when $4|x| < 1 \Rightarrow |x| < \frac{1}{4}$ $\boxed{R = \frac{1}{4}}$



Check endpoints

$\underline{x = \frac{1}{4}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n (\frac{1}{4})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

- Alternating ✓
- Decreasing ✓
- $\displaystyle\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

Convergent

$\underline{x = -\frac{1}{4}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n (-\frac{1}{4})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ divergent.

Interval of Convergence

$$\left(-\frac{1}{4}, \frac{1}{4}\right]$$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+1)^n}{3^{n+1}}$

Ratio Test

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)(x+1)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+1)^n} \right|$$

$$= \lim_{n \to \infty} \left| \left(\frac{n+1}{n}\right)(x+1)\left(\frac{1}{3}\right) \right|$$

$$= \frac{1}{3}|x+1|$$

- Convergent if $\frac{1}{3}|x+1| < 1 \Rightarrow |x+1| < 3$
- Divergent if $|x+1| > 3$

$\boxed{R = 3}$

$\underline{x=2}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{3} \to \infty$ diverges

$\underline{x=-4}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{3}$ $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n}{3}$ does not exist

diverges

Interval of convergence $(-4, 2)$

Representing Functions by Power Series

Recall $f(x) = \dfrac{1}{1+x} = 1 + x + x^2 + x^3 + \dots = \displaystyle\sum_{n=0}^{\infty} x^n \quad |x| < 1$

What about $\dfrac{1}{1+x^2}$?

$$g(x) = \frac{1}{1-(-x^2)} = f(-x^2) = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

Converges when $|x^2| < 1 \iff |x| < 1$

Examples

① Find a power series representation of $\frac{1}{x+3} = h(x)$

$$h(x) = \frac{1}{x+3} = \frac{1}{3\left(1+\frac{x}{3}\right)}$$

$$= \frac{1}{3\left(1-\left(-\frac{x}{3}\right)\right)}$$

$$= \frac{1}{3} f\left(-\frac{x}{3}\right)$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$$

Converges when $\left|-\frac{x}{3}\right| < 1$, or when $|x| < 3$. Interval of convergence is $(-3, 3)$.

② $g(x) = \frac{x^3}{x+3}$

$$g(x) = x^3\left(\frac{1}{x+3}\right)$$

$$= x^3\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{n+3}$$

$$= \frac{1}{3}x^3 - \frac{1}{9}x^4 + \frac{1}{27}x^5 - \frac{1}{81}x^7$$

$$= \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{3^{n-2}} x^n$$

Radius of convergence is still $(-3, 3)$