Lecture 17 Alternating Series, Absolute Convergence, & Ratio & Root Tests

So Far: Series with Positive Terms

Today: Series with Some Negative Terms

An alternating series is one whose terms are alternately positive or negative.

<u>Examples</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} + \dots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

Terms are $a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$ (here $b_n = |a_n|$).

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \qquad b_n > 0$$

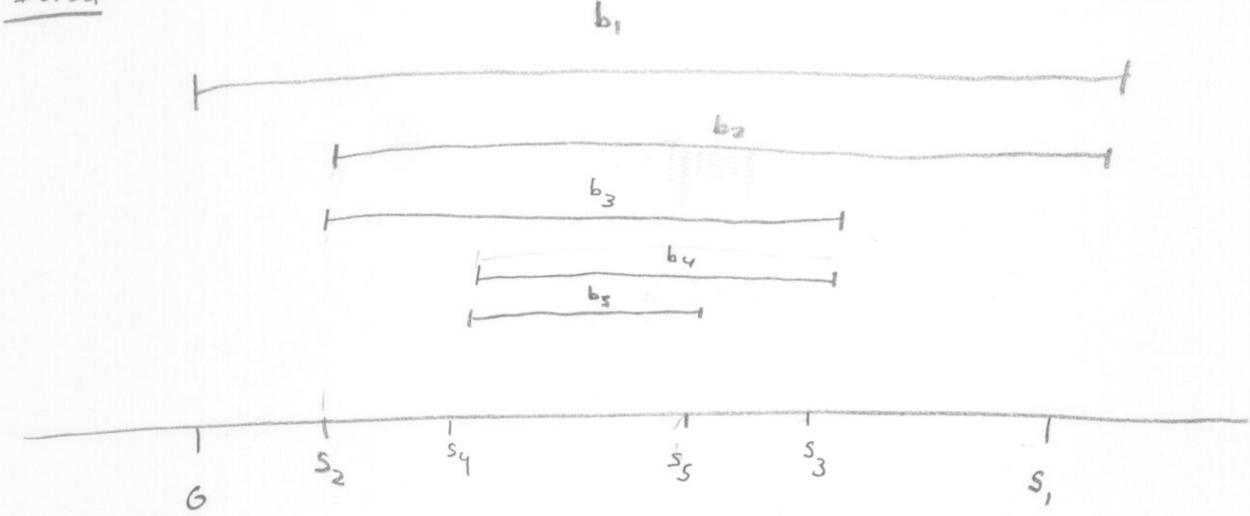
Satisfies

1. $b_{n+1} \le b_n$ for all n

2. $\lim_{n \to \infty} b_n = 0$

then the series converges.

Idea



Proof Consider even partial sums.

$$s_2 = b_1 - b_2 \geq 0 \qquad \text{since } b_2 \leq b_1$$

$$s_4 = (b_1 - b_2) + (b_3 - b_4) \geq s_2 \qquad \text{since } b_3 \geq b_4$$

$$s_5 = (b_1 - b_2) + (b_3 - b_4) + (b_5 - b_6) \geq s_4 \qquad \text{since } b_5 \geq b_6$$

So even partial sums are increasing

$$0 \leq s_2 \leq s_4 \leq s_6 \leq \dots \leq s_{2n} \leq \dots$$

Also

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots$$
$$< b_1$$

So even partial sums bounded & increasing, hence converge to some s.

What about odd

$$s_{2n+1} = s_{2n} + b_{2n+1}$$

$$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} s_{2n} + \lim_{n \to \infty} b_{2n+1}$$
$$= s + 0$$
$$= s$$

So $\sum (-1)^n b_n = s$

Examples

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

- Alternating ✓
- $b_{n+1} < b_n$ ✓ $\frac{1}{n+1} < \frac{1}{n}$
- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$

Convergent

2. $\sum_{n=1}^{\infty} \frac{(-1)^n 5n}{6n-1}$

- Alternating ✓
- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{5n}{6n-1} = 5/6 \neq 0$

Divergent

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

- Alternating ✓
- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n^2}{n^3+1} = 0$ ✓
- Decreasing? $f(x) = \frac{x^2}{x^3+1} = \frac{2x(x^3+1) - 3x^2(x^2)}{(x^3+1)^2}$

$$= \frac{x(2-x^3)}{(x^3+1)^2}$$

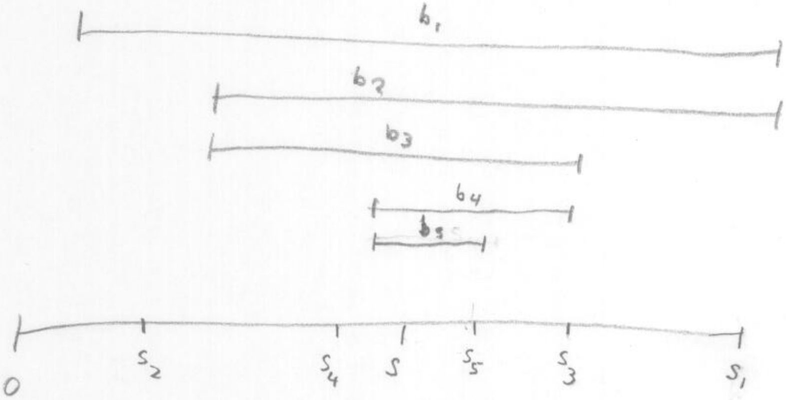
$$< 0 \quad \text{if} \quad x \geq \sqrt[3]{2}$$

Hence convergent.

Estimating Sums

$$s = \underbrace{s_n}_{a_1+\dots+a_n} + \underbrace{R_n}_{a_{n+1}+a_{n+2}+\dots}$$

Alternating Series Estimation Thm



Notice

$$S - s_4 \leq s_5$$

Alternating Series Estimation Thm

IF $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series satisfying

① $b_{n+1} \leq b_n$

② $\lim_{n \to \infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Proof For any even n, $s_n \leq s \leq s_{n+1}$. Ergo $|s - s_n| \leq |s_{n+1} - s_n| \leq b_{n+1}$. Similarly for n odd.

Example Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to three decimal places.

Want $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = s_n + R_n$

$.001 < R_n < .001$

- Alternating ✓
- $\frac{1}{n!} > \frac{1}{(n+1)!}$ ✓
- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n!} = 0$ ✓

Know $|R_n| < b_{n+1}$

$|R_n| < \frac{1}{(n+1)!}$

Want $|R_n| < \frac{1}{(n+1)!} < .001$

$\frac{1}{(n+1)!} < .001$

$1000 < (n+1)!$

$n+1 = 7$

$n = 6$

$s_6 = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$

$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$

$\approx .368$

Absolute Convergence & the Ratio & Root Tests

$\sum a_n$ $\sum |a_n|$

arbitrary terms   only positive terms (well understood)

<u>Defn</u> We say $\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ is convergent.

<u>Examples</u>

① $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent and absolutely convergent.

② $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent.

<u>Defn</u> A series $\sum a_n$ is called <u>conditionally convergent</u> if it is convergent but not absolutely convergent.

How are these things related?

<u>Thm</u> If a series $\sum a_n$ is absolutely convergent, then it is convergent.

<u>Proof</u>

Note

$$0 \le a_n + |a_n| \le 2|a_n| \quad \text{because } |a_n| = a_n \text{ or } -a_n$$

If $\sum |a_n|$ is convergent, so is $\sum 2|a_n|$. Ergo by comparison $\sum a_n + |a_n|$ is. But

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$ is a difference of two

convergent series, ergo convergent.

Examples

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

- Positive & negative terms, not alternating.
- Look at $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$. Positive terms, so can use comparison.
- $0 < \left|\frac{\sin(n)}{n^2}\right| < \frac{1}{n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

Two More Tests Related to Absolute Convergence

Mimic a geometric or p-series

The Ratio Test

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, ???

All conditional convergence falls here

Proof

Idea: Compare to a geometric series.

1. Since $L<1$, pick $L<r<1$. Since $\left|\frac{a_{n+1}}{a_n}\right|$ approaches L, there is N such that $n \geq N$ means $\left|\frac{a_{n+1}}{a_n}\right| < r$. So $|a_{n+1}| < r|a_n|$

- Then $r|a_N| > |a_{N+1}|$, $r^2|a_N| > r|a_{N+1}| > |a_{N+2}|$, etc.
- In general $|a_{N+k}| < r^k |a_N|$

$$\uparrow_{r} \quad \uparrow_{a}$$

The series $\sum_{k=1}^{\infty} |a_N| r^k$ is geometric & convergent, so

$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}|$ converges. Hence $\sum_{n=1}^{\infty} |a_n|$ is convergent.

2. If $\left|\frac{a_{n+1}}{a_n}\right| \to L > 1$ or to ∞, there is an N such that for $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| > 1$. Then $|a_{n+1}| > |a_n|$ for $n \geq N \Rightarrow \lim_{n\to\infty} a_n \neq 0$. So $\sum a_n$ diverges.

Examples

1. $$\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{4^n}$$

$$a_n = \frac{(-1)^n n^4}{4^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^4}{4^{n+1}}}{\frac{(-1)^n (n)^4}{4^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^4}{4(n^4)} \right|$$

$$= \frac{1}{4} < 1$$

So the series is absolutely convergent

2. $$\sum \frac{n^n}{n!}$$ (Note: $\frac{n!}{n^n}$ can be done by comparison - hw)

$$a_n = \frac{n^n}{n!} > 0$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{n^n}\right) \cdot \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Root Test This time we mimic a p-series.

① If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

② If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or is ∞, $\sum_{n=1}^{\infty} a_n$ diverges

③ If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, ???

Surprisingly useful - we'll see next time

Example

$$\sum_{n=1}^{\infty} \left(\frac{2n+4}{4n+3}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+4}{4n+3}\right)^n} = \lim_{n\to\infty} \frac{2n+4}{4n+3} = \frac{1}{2}$$

Convergent by Root Test

Rearrangements

Absolute convergence vs. conditional convergenge has something to do with what happens when we rearrange the terms.

Finite Sum

$$1+3+4+7 = 7+3+1+4$$

Absolutely Convergent Series

$$a_1 + a_2 + a_3 + a_4 + \cdots = a_5 + a_7 + a_1 + a_{200} + \cdots$$

If $\sum a_n$ is absolutely convergent with sum s, any rearrangement of $\sum a_n$ has sum s.

<u>However</u> (A Magic Trick)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2 \qquad (1)$$

$$\times \frac{1}{2} \qquad \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \dots = \frac{1}{2}\ln 2 \qquad (2)$$

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots = \frac{1}{2}\ln 2 \qquad (3)$$

Add (1) and (3)

$$\underbrace{1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4}}_{\text{Same terms as Series (1)}} + \dots = \frac{3}{2}\ln 2$$

If $\sum a_n$ is a conditionally convergent series and r is <u>any</u> real number, there is a rearrangement of $\sum a_n$ summing to r.