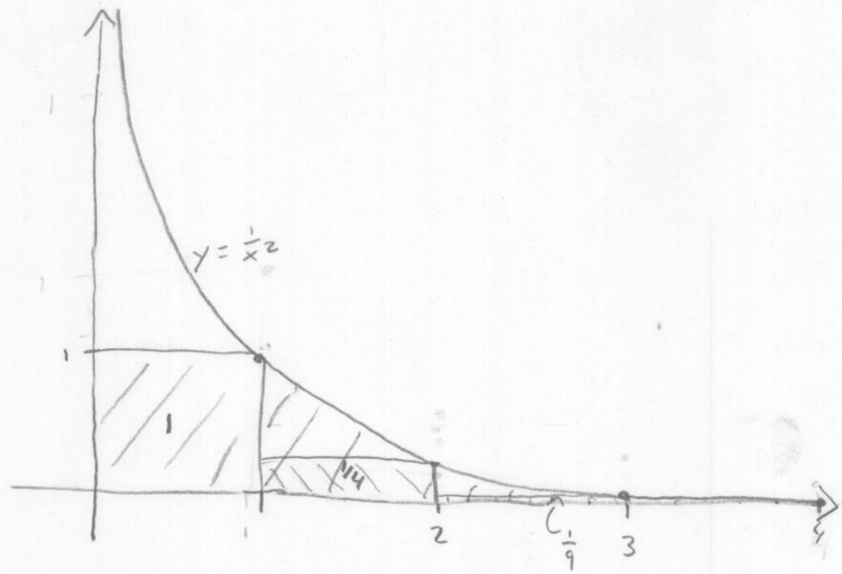
Example

(1) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$



Rectangles are the sum of the series

$$0 < \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \left(\frac{1}{4} + \frac{1}{9} + \dots\right)$$

$$< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

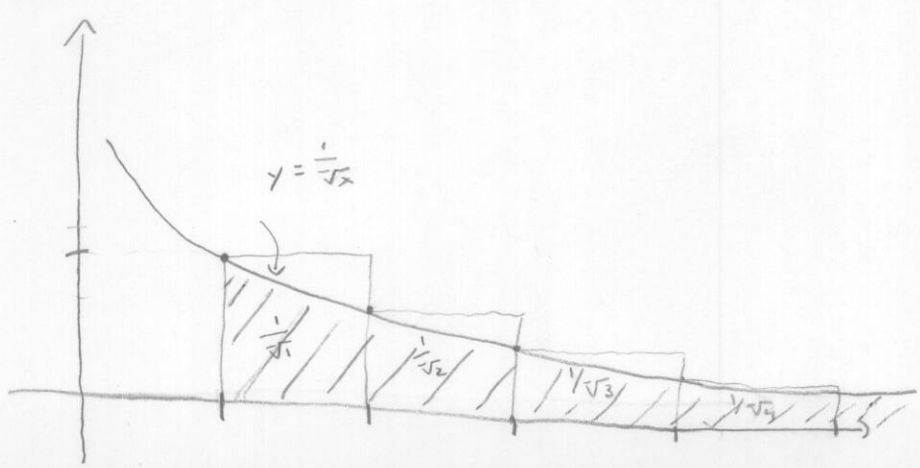
$$= 1 + 1$$

$$= 2$$

So $\displaystyle\sum_{i=1}^{\infty} \frac{1}{n^2}$ is finite

(Note: actally $\frac{\pi^2}{6}$)

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots$



Rectangles are sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

The Integral Test

Suppose f is cts, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if an only if $\int_1^{\infty} f(x)dx$ converges.

① If $\int_1^{\infty} f(x)dx$ converges, so does $\sum_{n=1}^{\infty} a_n$.

② If $\int_1^{\infty} f(x)dx$ diverges, so does $\sum_{n=1}^{\infty} a_n$.

Proof on next page

Some notes

ⓐ Don't have to start at 1. For $\sum_{n=N}^{\infty} a_n$ we test $\int_N^{\infty} f(x)dx$.

ⓑ f only has to be eventally decreasing (say on $[N, \infty)$). Then we know whether $\sum_{n=N}^{\infty} a_n$ converges, hence whether $\sum_{n=1}^{\infty} a_n$ does.

Examples

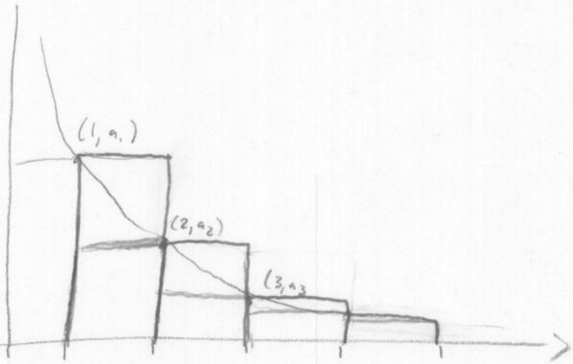
① $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

✓ Convergent.

$f(x) = \frac{1}{1+x^2}$ cts, positive, decreasing

$$\int_1^{\infty} \frac{1}{1+x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}dx = \lim_{t\to\infty} \tan^{-1}x\Big|_1^t = \lim_{t\to\infty} \tan^{-1}t - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Proof



$$a_2 + \dots + a_n \le \int_1^n f(x)\,dx \le a_1 + \dots + a_{n-1}$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent,

$$\sum_{i=2}^{n} a_i \le \int_1^n f(x)\,dx \le \int_1^\infty f(x)\,dx = M$$

So

$S_n = a_1 + \sum_{i=2}^{n} a_i < a_1 + M$ is bounded above by $a_n + M$, and below by 0.

As it is also monotone increasing, S_n converges $\Rightarrow \sum_{i=1}^{\infty} a_n$ converges.

(ii) If $\int_1^\infty f(x)\,dx$ diverges, $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$.

Since

$$\int_1^n f(x)\,dx \le \sum_{i=1}^{n-1} a_i = S_{n-1}, \quad S_{n-1} \to \infty \text{ as well}$$

2. When is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

$p < 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{-p} \longrightarrow \infty$ ← positive number

$p = 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} 1 \longrightarrow \infty$

$p < 0$ $\quad f(x) = \frac{1}{x^p}$ positive, cts, decreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ diverges if $p \leq 1$ from previous work

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ & divergent if $p \leq 1$.

<u>Propn</u> The <u>p-series</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

<u>Examples</u>

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges.

<u>Note</u> Just because the series & improper integral both converge doesn't mean they converge to the same thing

$\int_1^{\infty} \frac{1}{x^2} = 1 \qquad \sum_{n=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \qquad \int_1^{\infty} f(x)\,dx \neq \sum_{n=1}^{\infty} a_n$

Example $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$?

$f(x) = \frac{\ln x}{x}$ Obviously positive, cts

$f'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ on $x > e$ So f is decreasing on $[e, \infty)$

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\,dx$$

$$= \lim_{t \to \infty} \frac{(\ln x)^2}{2} \Big]_1^t$$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2}$$

$$= \infty$$

So the improper integral, and thus the sequence, is divergent.

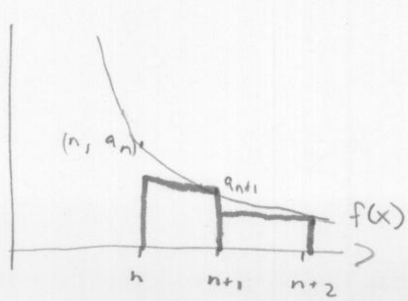
Error Bounds

The integral test also gives us an idea of how close a finite estimate of a convergent series is.

Let R_n be nth remainder, difference between actual sum and nth partial sum.

$$R_n = s - s_n$$

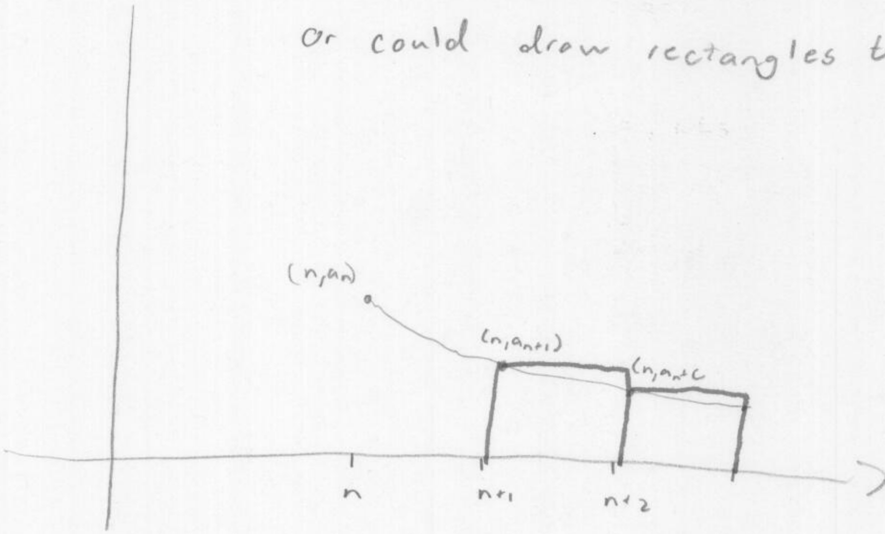
$$= a_{n+1} + a_{n+2} + \cdots$$



$$\int_n^{\infty} f(x)\,dx \geq R_n$$

Could draw rectangles to the left

Or could draw rectangles to the right



$$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$$

So

$$\boxed{\int_n^{\infty} f(x)\,dx \geq R_n \geq \int_{n+1}^{\infty} f(x)\,dx.}$$

↖ usually we care about the first one

Example

(a) Approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ using five terms. How accurate is the estimate?

$$S_5 = \frac{1}{1} + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625}$$

$$\approx 1.0804$$

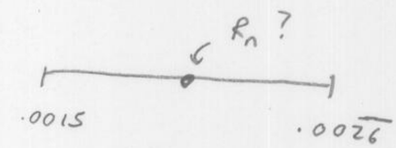
$$\int_5^{\infty} \frac{1}{x^4}\,dx \geq R_5 \geq \int_6^{\infty} \frac{1}{x^4}\,dx$$

$$\int_k^{\infty} \frac{dx}{x^4} = \lim_{t \to \infty} \left[\frac{-1}{3x^3}\right]_k^t = \lim_{t \to \infty} \left[\frac{-1}{3t^3} + \frac{1}{3k^3}\right] = \frac{1}{3k^3}$$

$$\frac{1}{3(5^3)} \geq R_5 \geq \frac{1}{3(6^3)}$$

$$.002\overline{6} \geq R_5 \geq .0015$$

Approximate by a midpoint.



$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1.0804 + R_n$$

$$\approx 1.0804 + \frac{.0026 + .0015}{2}$$

$$= 1.0804 + .0021$$

$$= 1.0825$$

(b) How large a partial sum do we have to take to have error $R_n < .0001$?

Want

Interpose a computable term in the inequality.

$$R_n \leq \int_n^{\infty} \frac{dx}{x^4} < .0001$$

$$\frac{1}{3n^3} < .0001$$

$$\frac{10000}{3} < n^3$$

$$n = 14.9$$

Need at least 15 terms.

The Comparison Test

Example

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

Looks like $\frac{1}{2^n}$, which we understand.

$$\frac{1}{2^n+1} < \frac{1}{2^n} \quad \text{for all } n.$$

Partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ will be less than partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and also positive, so $0 \le \sum_{n=1}^{\infty} \frac{1}{2^n+1} < 1$. Should converge.

Formally

The Comparison Test

Let $\sum a_n$, $\sum b_n$ be series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, $\sum a_n$ is convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, $\sum a_n$ is divergent.

Proof (i) Let $s_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=1}^{n} b_n$, $t = \sum_{n=1}^{\infty} b_n$.

- Since both sequences have positive terms, $\{s_n\}$ and $\{t_n\}$ are increasing. Since $t_n \to t$, $t_n \le t$ for all n.

  Moreover, $s_n \le t_n \le t$ for all n, since $a_n \le b_n$. Ergo $\{s_n\}$ is increasing & bdd above, hence converges by monotone sequence theorem. So $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, $t_n \to \infty$ (since $\{t_n\}$ is increasing. But $a_n \geq b_n \Rightarrow s_n \geq t_n \Rightarrow s_n \to \infty$.

So $\sum a_n$ diverges.

Popular comparisons

- A p-series $\sum \frac{1}{n^p}$. Convergent $\Leftrightarrow p > 1$
- A geometric series $\sum ar^{n-1}$ Convergent $\Leftrightarrow |r| < 1$

Examples

(1) $\sum_{n=1}^{\infty} \frac{7}{3n^2+4n+5}$

- $0 < \frac{7}{3n^2+4n+5} < \frac{7}{3n^2} = \frac{7}{3}\left(\frac{1}{n^2}\right)$
- $\sum_{n=1}^{\infty} \frac{7}{3}\left(\frac{1}{n^2}\right)$ converges
- $\sum_{n=1}^{\infty} \frac{7}{3n^2+4n+5}$ converges.

Note Of course we only need to show this holds on $n \geq N$; that is, eventually, since $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=N}^{\infty} a_n$ converges.

(2) $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$. Could use integral test

- Alternately
  - $\frac{\ln k}{k} > \frac{1}{k} > 0$ when $k \geq 3$
  - $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges
  - $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ diverges.

Often straight up comparison doesn't work, but we feel like it should.

$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$? Not less than $\frac{1}{2^n}$, but approximates it closely.

We use an even stronger statement.

__The Limit Comparison Test__

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ for $0 < c < \infty$ then either both series converge or both diverge.

__Proof__ Let $m, M > 0$ such that $m < c < M$. There is an integer N such that for $n > N$, $m < \frac{a_n}{b_n} < M$

So for $n > N$, $mb_n < a_n < Mb_n$

If $\sum b_n$ converges, so does $\sum Mb_n \Rightarrow \sum a_n$ converges.
If $\sum b_n$ diverges, so does $\sum mb_n \Rightarrow \sum a_n$ diverges.

Examples

① $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. Compare to $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$a_n = \frac{1}{2^n-1}$ $\quad b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0 \quad \checkmark$$

So $\sum \frac{1}{2^n-1}$ converges.

② $\sum_{n=1}^{\infty} \frac{3n^2+4n}{\sqrt{2+n^5}}$ ← degree 2, ← degree 3/2 $\quad$ Get rid of lower order terms

$a_n = \frac{3n^2+4n}{\sqrt{2+n^5}}$ $\quad b_n = \frac{3n^2}{\sqrt{n^5}} = \frac{3}{n^{1/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2+4n}{\sqrt{2+n^5}} \cdot \frac{n^{1/2}}{3}$$ ← degree 5/2, ← degree 5/2

$$= \frac{3}{3}$$

$$= 1$$

Estimating Sums

Suppose we compared $\overset{s}{\overset{\shortparallel}{\sum a_n}}$ with $\overset{t}{\overset{\shortparallel}{\sum b_n}}$, and we can estimate the remainder of $\sum b_n$

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \dots$$

<u>Know</u> $a_n \leq b_n$ for all n

<u>So</u> $R_n \leq T_n$ IF $\sum b_n$ is a p-series or a geometric series we can estimate this error.

<u>Example</u>

Estimate the error involved in approximating $\sum \frac{1}{n^4+1}$ by the hundredth partial sum.

$$\frac{1}{n^4+1} < \frac{1}{n^4}$$

$$T_n \leq \int_n^\infty \frac{1}{x^4}\,dx = \frac{1}{3n^3}$$

$$T_{100} \leq .3.3 \times 10^{-7}$$

$$R_{100} \leq T_{100} \leq .3.3 \times 10^{-7}$$