Lecture 15

What is a series? The sum of a sequence.

Example

Sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ $\qquad \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

Series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ $\qquad \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\underbrace{\underbrace{\underbrace{\underbrace{\tfrac{1}{2}}_{\tfrac{1}{2}} + \tfrac{1}{4}}_{3/4} + \tfrac{1}{8}}_{7/8} + \tfrac{1}{16}}_{15/16}$$

Sum seems to be approaching 1.

In general Use finite sums to approximate infinite sums (analogous to using finite integrals to approximate infinite integrals, only more flexible, since addition is easier than integration).

Consider partial sums

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = \sum_{i=1}^{n} a_i$$

$\{s_n\}_{n=1}^{\infty}$ is a sequence. It may or may not have a limit.

IF $\lim_{n\to\infty} s_n$ exists, it is the sum of the sequence.

Defn Given $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$, and $s_n = \sum_{i=1}^{n} a_i = a_1 + \cdots + a_n$,

if $\{s_n\}$ is convergent with $\lim_{n \to \infty} s_n = s$, we say $\sum_{i=0}^{\infty} s_n$ is convergent and $\sum_{n=1}^{\infty} a_n = s$. This number is the <u>sum</u> of the series. If $\{s_n\}$ is divergent, the series is divergent.

<u>Example</u>

$$\sum_{i=1}^{n} \frac{1}{2^n} = 1$$

$$s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \qquad \lim_{n \to \infty} 1 - \frac{1}{2^n} = 1$$

$$= \frac{2^{n-1} + 2^{n-2} + \cdots + 1}{2^n}$$

$$= \frac{2^n - 1}{2^n}$$

$$= 1 - \frac{1}{2^n}$$

<u>Example</u> <u>Geometric Series</u>

$$\sum_{n=0}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \qquad a \neq 0$$

Each term obtained from previous by multiplying by a <u>common ratio</u> r.

<u>$r=1$</u> $s_n = a + \cdots + a = na \longrightarrow \infty$ diverges

If $r \neq 1$,
$$s_n = a + ar + \cdots + ar^n$$
$$- \quad rs_n = \quad ar + \cdots + ar^n + ar^{n-1}$$
$$= a - ar^{n-1}$$

So

$$s_n - r s_n = a - ar^{n+1}$$

$$(1-r)s_n = a - ar^{n+1}$$

$$s_n = \frac{a(1-r^{n+1})}{1-r}$$

If $-1 < r < 1$, $r^{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$

so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}$

$$= \frac{a}{1-r}$$

If $r \le -1$ or $r > 1$

$r^{n+1} \longrightarrow \infty$,

so the sequence diverges.

Thm The geometric series

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

is convergent if $|r| < 1$ and its sum is $\frac{a}{1-r}$.

If $|r| \ge 1$, the geometric series is divergent.

Examples ① $7 - \frac{14}{3} + \frac{28}{9} - \frac{56}{27} + \ldots$

$a = 7$

$r = \frac{-2}{3}$

$$\sum_{n=0}^{\infty} 7\left(-\tfrac{2}{3}\right)^n = \frac{7}{1+\frac{2}{3}} = \frac{7}{5/3} = \boxed{\frac{21}{5}}$$

② $$\sum_{n=1}^{\infty} 3^{2n} 2^{1-n} = \sum_{n=1}^{\infty} (3^2)^n 2^{-(n-1)} = \sum_{n=1}^{\infty} 9\left(\frac{9}{2}\right)^{n-1}$$

$r = \frac{9}{2} > 1$

Series diverges!

(3) Write $4.3\overline{18}$ repeating as a fraction.

$$4.3 + \frac{18}{10^3} + \frac{18}{10^5} + \frac{18}{10^7} + \cdots \qquad a = \frac{18}{10^3} \quad r = \frac{1}{10^2}$$

$$4.3\overline{18} = 4.3 + \frac{\frac{18}{10^3}}{1 - \frac{1}{10^2}}$$

$$= 4.3 + \frac{\frac{18}{1000}}{\frac{99}{100}}$$

$$= \frac{43}{10} + \frac{18}{990}$$

$$= \frac{4275}{990}$$

$$= \frac{855}{198}$$

$$= \frac{95}{22}$$

(4) $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4, \quad \text{For } |x| < 1$$

$$= \frac{1}{1-x}$$

Another easy to compute case: <u>Telescoping Series</u>

<u>Example</u>

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \qquad \text{Hmm}\ldots$$

Use partial Fractions!

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1}$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right) + \dots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{n+1}$$

$$= 1$$

$$\sum_{i=1}^{\infty} \frac{1}{n(n+1)} = \boxed{1}$$

We say a sequence of the Form $\sum_{i=1}^{n} f(n) - f(n+1)$ is <u>telescoping</u>.

<u>Example</u> The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$ We use a trick

Consider S_{2^n} partial sums.

$$S_2 = 1 + \frac{1}{2} = 3/2$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 2\frac{1}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$> 3$$

In general $s_{2^n} > 1 + \frac{n}{2}$. So $s_n \longrightarrow \infty$ as $n \to \infty \Rightarrow$ harmonic series diverges.

Conditions on Convergence

Thm IF $\sum_{n=1}^{\infty} a_n$ exists, $\lim_{n\to\infty} a_n = 0$. (Test for Divergence)

Proof Let $s_n = a_1 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Let $\lim_{n\to\infty} s_n = s$.
Then $\lim_{n\to\infty} s_{n-1} = s$ as well.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

$\square$

Warning Converse is not true! Consider the harmonic sequence and series!

Note Two sequences $\{a_n\}$ - original sequence

$\{s_n\}$ - partial sums

$\{s_n\} \longrightarrow s$ implies $a_n \longrightarrow 0$, $\sum a_n = s$

Examples

① $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{6n^3+7}$

$$\lim_{n\to\infty} \frac{n^3}{6n^3+7} = \frac{1}{6}$$, so sequence diverges.

Further Properties (From Limit Laws for sequences)

Thm If $\sum a_n$, $\sum b_n$ are convergent series, so are $\sum c a_n$, $\sum (a_n + b_n)$, $\sum (a_n - b_n)$ and

(i) $\displaystyle\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example of proof

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Partial sums of $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n)$ are $\displaystyle u_n = \sum_{i=1}^{n} (a_i + b_i) = s_n + t_n$

$$\begin{aligned}
\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{n\to\infty} u_n &= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i \\
&= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \\
&= s + t
\end{aligned}$$

Example

$$\sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{3^n} \quad \text{if both exist}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$= 4(1) + \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= 4 + \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= 4 + \frac{1}{2}$$

Note Beginning terms aren't important.

If $\sum_{n=4}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

Example $47, 85, 102, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ converges.

Often we're only interested in whether a series converges to a sum or not.

For sequences we used what we know about limits of functions at ∞. For series we should use what we know about integrals at ∞