Lecture 14

What is a sequence? A list of numbers.

$$\{a_1, a_2, a_3, a_4, a_5, a_6, \dots\}$$

a_1 - First term

a_2 - second term

a_3 - third term

$\vdots$

Also written

$\{a_n\}$ or $\{a_n\}_{n=i}^{\infty}$

Examples

(a) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\}$ $\quad a_n = \frac{1}{n}$ $\quad \{\frac{1}{n}\}_{n=1}^{\infty}$

(b) $\{\frac{-3}{2}, \frac{4}{4}, \frac{-5}{8}, \frac{6}{16}, \dots\}$ $\quad a_n = \frac{(-1)^n(n+2)}{2^n}$ $\quad \left\{\frac{(-1)^n(n+2)}{2^n}\right\}_{n=1}^{\infty}$

(c) $\{2, \sqrt{5}, \sqrt{6}, \sqrt{7}, 3, \dots\}$ $\quad a_n = \sqrt{n-1}$ $\quad \{\sqrt{n-1}\}_{n=5}^{\infty}$

(d) $\{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \dots\}$ $\quad a_n = \sin\left(\frac{n\pi}{6}\right)$ $\quad \{\sin\left(\frac{n\pi}{6}\right)\}_{n=0}^{\infty}$

Note that n doesn't have to start at 1.

Example

What sequence is

$a_1 = \frac{2}{3}$ $a_2 = \frac{-3}{9}$ $a_3 = \frac{4}{27}$ $a_4 = \frac{-5}{81}$ $a_5 = \frac{6}{243}$ ...

$$a_n = \frac{(-1)^{n+1}(n+1)}{3^n} \qquad \left\{\frac{(-1)^{n+1}(n+1)}{3^n}\right\}_{n=1}^{\infty}$$

Not every sequence has a defining equation; some involve nonalgebraic data

Example

① $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ Fibonacci sequence

Define recursively
$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

② $\{3, 1, 4, 1, 5, 9, 2, 6, 5, \ldots\}$ Digits of pi

$\uparrow$
a_0

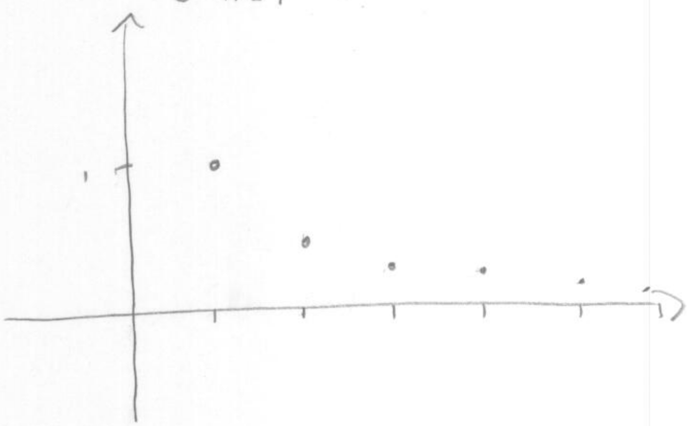
a_n = number in the nth place

③ $\{59, 50, 42, 34, 28, 23, 18, 14, \ldots\}$

What comes next? Christopher St. These are the stops on the ① train below Columbus Circle.

What can we say about the behavior of a sequence $\{a_n\}$ as $n \to \infty$?

<u>Example</u> $\begin{Bmatrix} a_n = \frac{1}{n} \\ n \geq 1 \end{Bmatrix}$ Plot points (n, a_n) on a coordinate axis



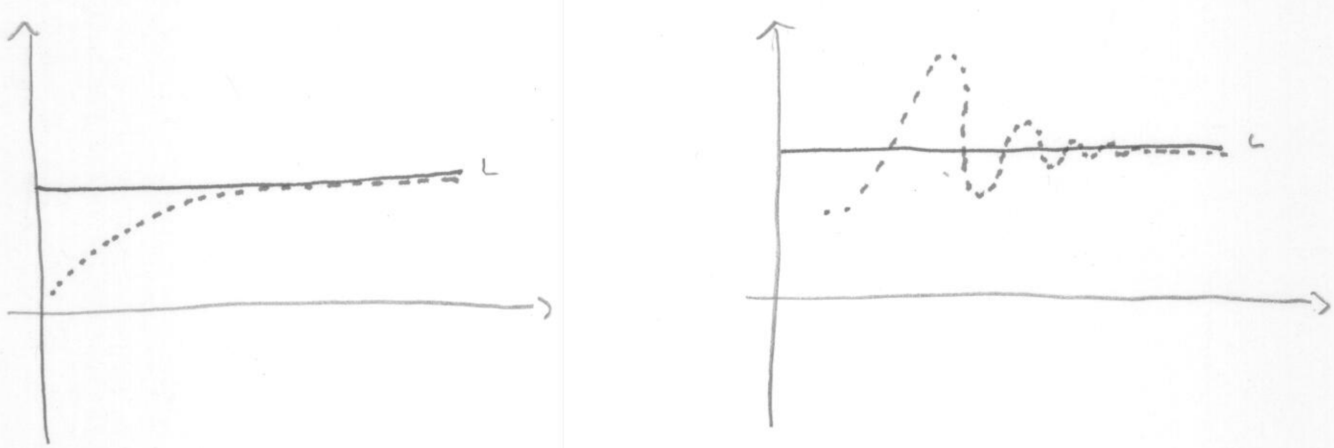
As n becomes large, $\frac{1}{n} \to 0$.

So it's reasonable to talk about a sequence having a limit.

<u>Defn</u> A sequence $\{a_n\}$ has the <u>limit</u> L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If the limit exists, we say the series <u>converges</u>; if not we say it <u>diverges</u>



Two sequences with limit L.

<u>Alternately</u>

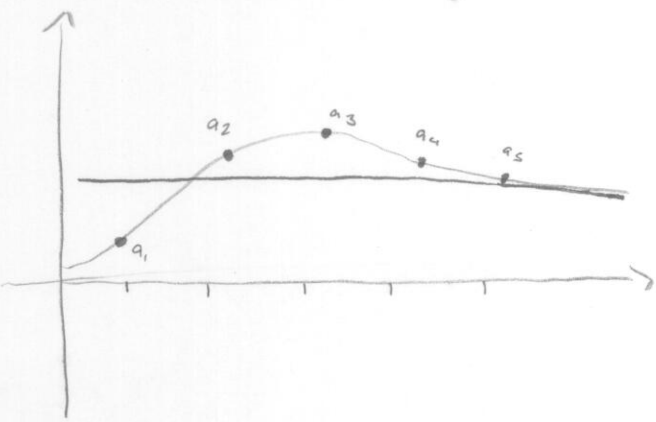
<u>Defn</u> A sequence a_n has limit L if $\forall\, \varepsilon > 0$ there is an N st if $n > N$ then $|a_n - L| < \varepsilon$.

This looks almost identical to our previous definition of the limit of a function. We should exploit this!

$\lim_{n \to \infty} a_n = L$ $\qquad$ $\lim_{x \to \infty} f(x) = L$

only difference is x can be any number, but n has to be an integer

"x is cts, n is discrete"



Suppose $\exists$ f st $f(n) = a_n$.

<u>Thm</u> If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n \to \infty} a_n = L$.

<u>Examples</u>

① $a_n = \dfrac{n}{n+1}$ $\qquad$ $\lim_{n \to \infty} \dfrac{n}{n+1} = \lim_{x \to \infty} \dfrac{x}{x+1} = \boxed{1}$

$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$

② $a_n = \dfrac{1}{n^r}$ $\quad r > 0$ $\qquad$ $\lim_{n \to \infty} \dfrac{1}{n^r} = \lim_{x \to \infty} \dfrac{1}{x^r} = \boxed{0}$

$\left\{ \frac{1}{1}, \frac{1}{2^r}, \frac{1}{3^r}, \ldots \right\}$

We also have the notion of an infinite limit.

<u>Defn</u> $\lim_{n \to \infty} a_n = \infty$ means that for every positive number M $\exists$ an integer N such that if $n > N$, $a_n > M$.

This is a special case of divergent sequences, in which the limit gets arbitrarily large.

Example

$\{1, 2, 3, 4, 5, 6, \ldots\}$ $\quad a_n = n$ $\quad$ As $n \to \infty$, $a_n \to \infty$

All the limit laws still hold by the same proofs as for functions.

If $\{a_n\}$, $\{b_n\}$ are convergent, and $c \in \mathbb{R}$,

1. $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$

2. $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$

3. $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n$

4. $\lim_{n \to \infty} c = c$

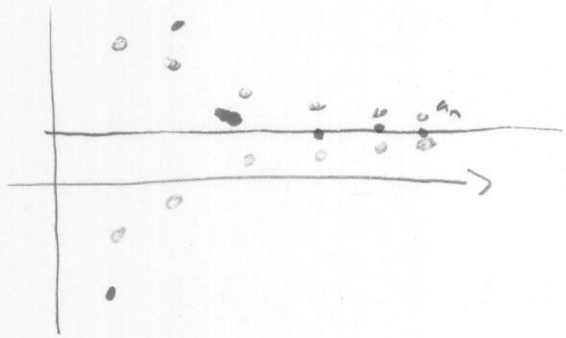
5. $\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$

6. $\lim_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$ if $\lim_{n \to \infty} b_n \neq 0$

7. $\lim_{n \to \infty} a_n^p = \left[\lim_{n \to \infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

We also still have the Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, $\lim_{n\to\infty} b_n = L$.



One more useful fact.

Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples Convergent or divergent?

① $a_n = \dfrac{n}{\sqrt{5+n}}$

$$\lim_{n\to\infty} \frac{n}{\sqrt{5+n}} = \lim_{n\to\infty} \frac{\overset{\rightsquigarrow 1}{1}}{\underbrace{\sqrt{\frac{5}{n^2}+\frac{1}{n}}}_{\rightsquigarrow 0}} = \infty$$

diverges to ∞

2) $a_n = \frac{\ln(n)}{n}$

Replace w/ a Function

$$\lim_{n \to \infty} \frac{\ln(n)}{n} = \lim_{x \to \infty} \frac{\ln(x)^{\rightsquigarrow \infty}}{x_{\rightsquigarrow \infty}}$$

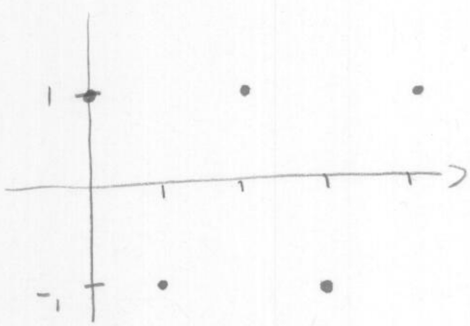
$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

Can't apply L'Hospital here because it pertains to Functions

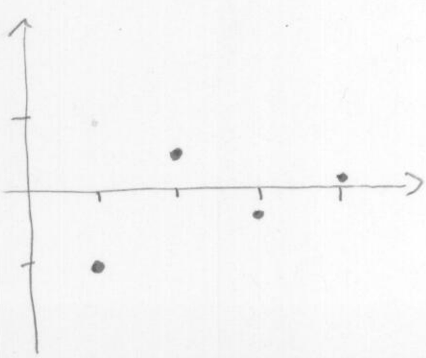
3) $a_n = (-1)^n, n \geq 0$

$\{1, -1, 1, -1, 1, -1, \ldots\}$ diverges!



4) $a_n = \frac{(-1)^n}{n}$

$n \geq 1$



Take the limit of the absolute value.

$$\lim_{n \to \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \to \infty} \frac{1}{n} = 0$$

Therefore $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$ as well.

Note This only works if the limit is actually 0!

Notice If the sequence is partly constructed from cts ftns, we can use what we know about limits of cts functions to evaluate limits.

Theorem

If $\lim_{n\to\infty} a_n = L$ and f is cts at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example① $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$

$= \cos(0)$

$= 1$

Example② What about $a_n = \frac{n!}{n^n}$?

$a_1 = 1$ $\quad a_2 = \frac{1}{2}\cdot\frac{2}{2}$ $\quad a_3 = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3}{3}$

$$a_n = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{n\cdot n\cdot n \ldots n}$$

$$= \frac{1}{n}\underbrace{\left(\frac{2}{n}\cdot \ldots \frac{n}{n}\right)}_{\leq 1}$$

We see

$0 \leq a_n \leq \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ by the squeeze theorem.

Example 3

$a_n = r^n$ For what values of r does the series diverge?

If $-1 < r < 1$ $\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0$

If $r > 1$ $\lim_{n \to \infty} r^n = \infty$

If $r < -1$ $|r| \to \infty$

sign of r varies divergent

If $r = 1$ $\lim_{n \to \infty} 1^n = 1$

If $r = -1$ diverges

The sequence $a_n = r^n$ converges if $-1 \le r \le 1$ and $\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$

Some useful properties of sequences

Defn A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$, so that $a_1 < a_2 < a_3 < \ldots$. It is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Examples ① $\left\{\frac{2}{n+6}\right\}$ is decreasing

$$\frac{2}{n+6} > \frac{2}{(n+1)+6} = \frac{2}{n+7}$$

So $a_n > a_{n+1}$ for $n \geq 1$

② Show $a_n = \frac{n}{n^2+1}$ is decreasing.

Want $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \iff (n^2+1)(n+1) < n\left[(n+1)^2+1\right]$

$$\cancel{n^3} + \cancel{n^2} + \cancel{n} + 1 < n^3 + \cancel{2n^2} + \cancel{2n}$$

$$1 < n^3 + n^2 \quad \checkmark \text{ True for}$$

OR $f(x) = \frac{x}{x^2+1}$ has $f(n) = a_n$

$$f'(x) = \frac{1(x^2+1) - 2x(x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \text{ for } x > 1.$$ So f is decreasing, implying that the sequence is as well.

Sequences can also be bounded.

Defn A sequence $\{a_n\}$ is bounded above if there is a number M st $a_n \leq M$ for all $n \geq 1$. It is bounded below if there is an m st $m \leq a_n$ for all $n \geq 1$.

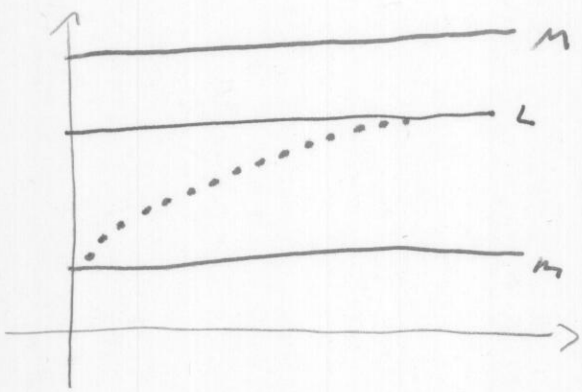
If $\{a_n\}$ is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Examples

$\{a_n = n\}$ bounded below but not above

$\{a_n = \frac{n}{n+1}\}$ bounded $0 < a_n < 1$

Suppose a sequence is both bounded and monotonic.



The sequence keeps increasing but runs out below M. So it must converge to a limit before that.

Thm Every montone bounded sequence is convergent.

Proof The real numbers have a property called the least upper bound property: any set $S \subseteq \mathbb{R}$ which is bounded has some b such that $s < L$ for all $s \in S$ and there is no $d < b$ with the same property.

Let $\{a_n\}$ be bounded and monotone increasing. Then for any $\varepsilon > 0$ there is some N st $L - \varepsilon < a_N < L$. But a_N is increasing, so $|L - a_n| < \varepsilon$ for all $n > N$. Ergo L is in fact the limit of the sequence.

Similarly for decreasing sequences and greatest lower bounds