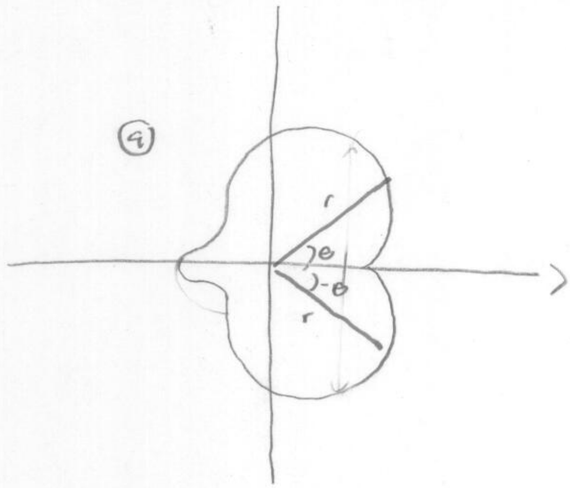
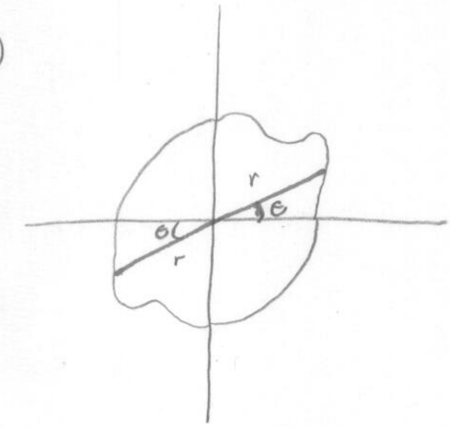
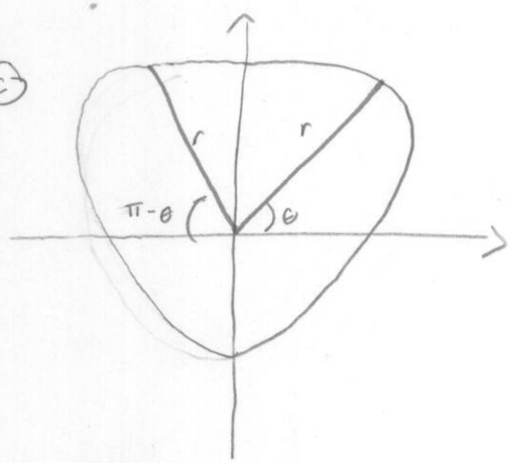






Lecture 13

Tangents

Suppose $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Both functions of a parameter θ

Use what we learned about general parametric equations.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Note: If we happen to be at the pole $(r=0)$ this reduces to

$$\frac{dy}{dx} = \tan\theta$$

Example

Back to our cardioid. Tangent line at $\theta = \frac{\pi}{4}$?

$(x,y) = (r\cos\theta,\ r\sin\theta)$

$r = 1+\sin\theta$

$$\frac{dr}{d\theta} = \cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-\sin\theta) - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

At $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{2}}\left(1+\frac{2}{\sqrt{2}}\right)}{\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{2}{\sqrt{2}}\right)}$$

$$= \left(\frac{\frac{1}{\sqrt{2}}+1}{1+\frac{1}{\sqrt{2}}-\frac{2}{\sqrt{2}}-1}\right)$$

$$= \frac{\frac{1+\sqrt{2}}{\sqrt{2}}}{\frac{-1}{\sqrt{2}}}$$

$$= -(1+\sqrt{2})$$

$x = \left(1+\sin\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$

$x = \frac{\sqrt{2}+1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)$

$= 1+\sqrt{2}$

$y = \left(1+\sin\pi/4\right)\sin\pi/4$

$= 1+\sqrt{2}$

$$\left(y - 1+\sqrt{2}\right) = -(1+\sqrt{2})\left(x - (1+\sqrt{2})\right)$$

Horizontal & vertical tangents?

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$\cos\theta = 0 \qquad \sin\theta = \frac{-1}{2}$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$\sin\theta = {}^{-}1 \qquad \sin\theta = \frac{1}{2}$

Horizontal

$\frac{\pi}{2} \quad \left(2, \frac{\pi}{2}\right)$

$\frac{7\pi}{6} \quad \left(\frac{1}{2}, \frac{7\pi}{6}\right)$

$\frac{11\pi}{6} \quad \left(\frac{1}{2}, \frac{11\pi}{6}\right)$

Vertical

$\frac{\pi}{6} \quad \left(\frac{3}{2}, \frac{\pi}{6}\right)$

$\frac{5\pi}{6} \quad \left(\frac{3}{2}, \frac{5\pi}{6}\right)$

$\frac{3\pi}{2} \quad (0,0) \quad ?$

$$\lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{dy}{dx} = \left(\lim_{\theta \to \frac{3\pi}{2}^-} \frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{\cos\theta \to 0}{1+\sin\theta \to 0}\right)$$

$$= \frac{-1}{3} \cdot \lim_{\theta \to \left(\frac{3\pi}{2}\right)^-} \frac{-\sin\theta \to {}^{-}1}{\cos\theta \to 0}$$

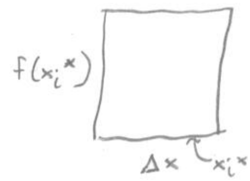
$$= \infty$$

Similarly $\lim_{\theta \to (3\pi/2)^+} \frac{dy}{dx} = -\infty$

Lecture 13

Areas & Lengths in Polar Coordinates

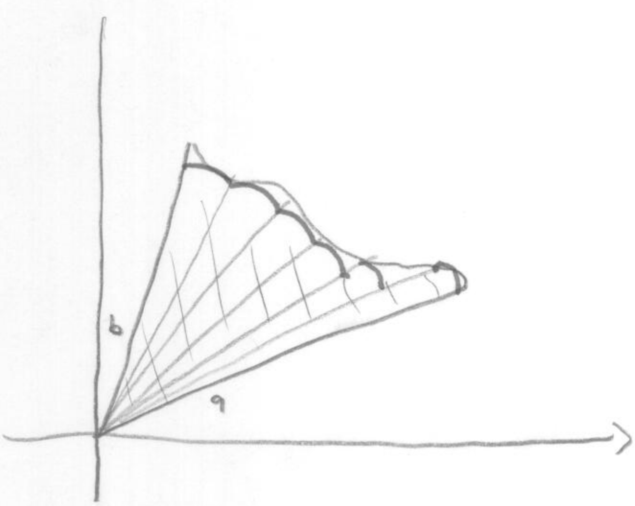
- Cartesian coordinates: Unit of area is a rectangle



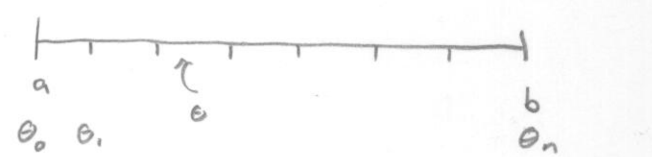
- Polar coordinates: Unit of area is a wedge of a circle.

$$A = \frac{\theta}{2\pi} \cdot (\pi r^2)$$
$$= \frac{1}{2}\theta r^2$$

Suppose R is a region bounded by $r = f(\theta)$, $\theta = a$, $\theta = b$.



Divide $[a, b]$ into n subintervals



$$\Delta\theta = \frac{b-a}{n} = \theta_i - \theta_{i-1}$$

Choose a $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Approximate by a wedge of radius $r_i^* = f(\theta_i^*)$

Area of wedge $= \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$

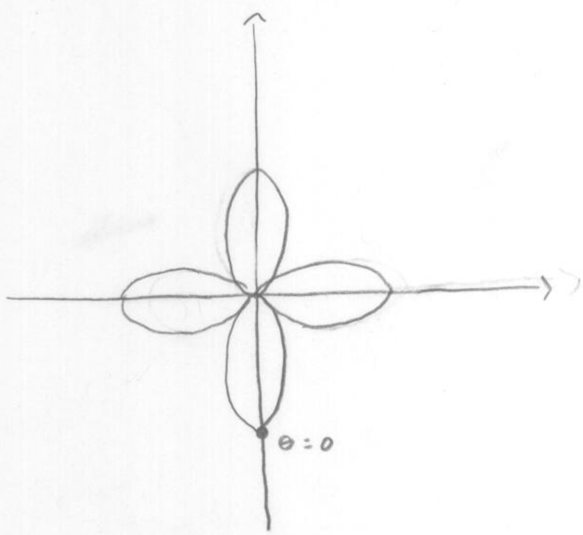
Total area $\approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$

Area of region $= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$

$$= \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

Example 1 Area enclosed by a loop of the four-leaved rose.

$r = \cos(2\theta)$



A single loop is traced $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2(2\theta)\, d\theta$$

$$= 2\int_0^{\pi/4} \frac{1}{2}\left(\frac{1}{2}(1+\cos(4\theta))\right) d\theta$$

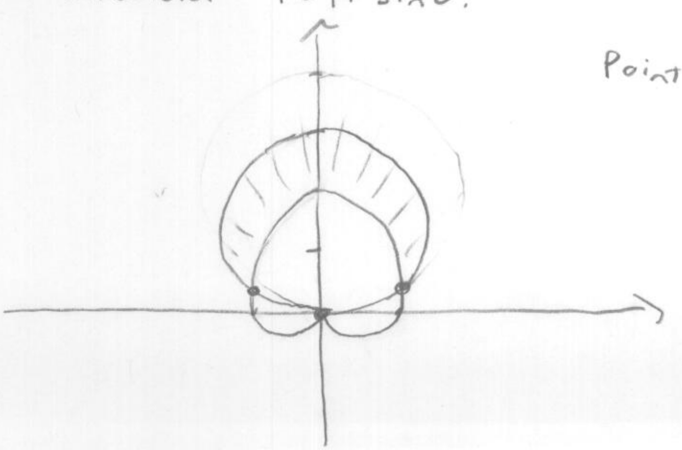
$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\left[\frac{\pi}{4} + \frac{1}{4}(0)\right] - \left[0+0\right]\right)$$

$$= \frac{\pi}{8}$$

Example 2

Find the area of the region inside the circle $r = 4\sin\theta$ 3 outside the cardioid $r = 1+\sin\theta$.



Points of intersection

$$1+\sin\theta = 3\sin\theta$$

$$1 = 2\sin\theta$$

$$\frac{1}{2} = \sin\theta$$

$$\theta = \pi/6,\ \frac{5\pi}{6}$$

Warning Note that the origin is a point of intersection we didn't pick up this way! This is because of the ambiguity in polar coordinates,

$r = 3\sin\theta$ $\quad$ $r = 1 + \sin\theta$

$(0,0)$ $(0,\pi)$ $\quad$ $(0, \frac{3\pi}{2})$

We pass through the origin at different θ values. (Different "times" if we think of θ as a parameter)

Always graph the curves to find all intersections.

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[3\sin\theta\right]^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} (1+\sin\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta\, d\theta - \int_{\pi/6}^{\pi/2} \left[1 + 2\sin\theta + \sin^2\theta\right] d\theta$$

Symmetric about $\theta = \frac{\pi}{2}$

$$= \int_{\frac{\pi}{6}}^{\pi/2} \left[8\sin^2\theta - 2\sin\theta - 1\right] d\theta$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} 8\left[\frac{1}{2}(1 - \cos 2\theta)\right) - 2\sin\theta - 1\Big] d\theta$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta$$

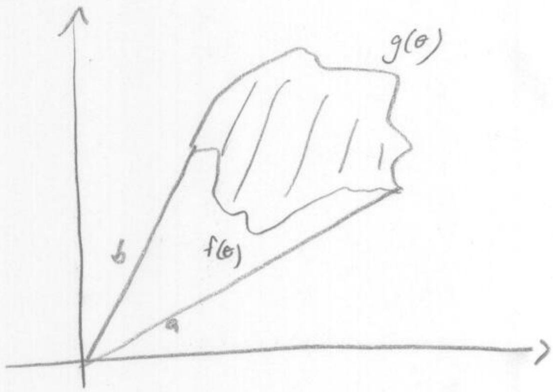
$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\pi/2}$$

$$= \left(\left[\frac{3\pi}{2} - 0 + 0\right] - \left[\frac{3\pi}{6} - 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2}\right]\right)$$

$$= \pi$$

In general

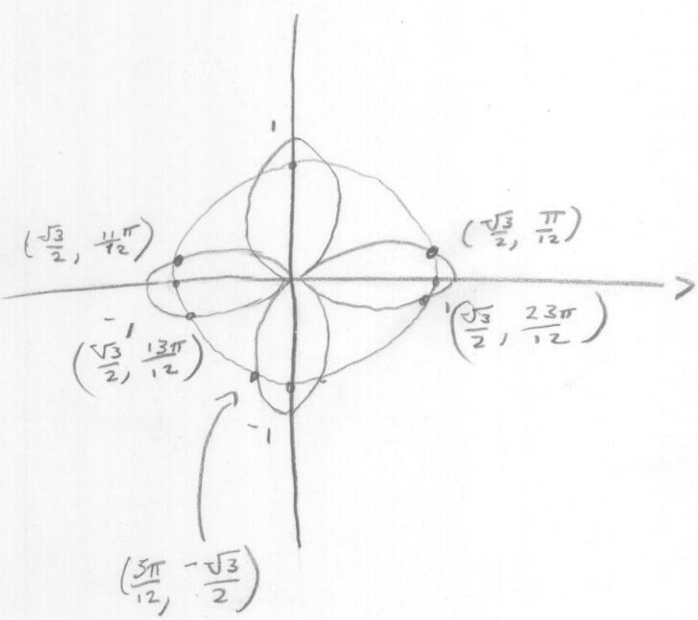
$$A = \int_a^b \frac{1}{2}\left(\left[f(\theta)\right]^2 - \left[g(\theta)\right]^2\right)\, d\theta$$



<u>Example</u> Intersection points.

$r = \cos 2\theta$ $\qquad r = \frac{\sqrt{3}}{2}$

$0 \le \theta \le 2\pi$



$\cos 2\theta = \frac{\sqrt{3}}{2} \qquad 0 \le 2\theta \le 4\pi$

$2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

But we only get 4 points!

<u>Note</u> $\cos 2\theta = -\frac{\sqrt{3}}{2}$ also lies on the circle.

$2\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$

$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

Arc length of a polar curve

$r = f(\theta) \quad a \le \theta \le b$ Use what we know about parametric equations and arc length.

- Regard θ as a parameter
- Parametric equations $\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

We know

$$L = \int_a^b \sqrt{\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2}\, d\theta$$

$$\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 = \left[\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta\right]$$

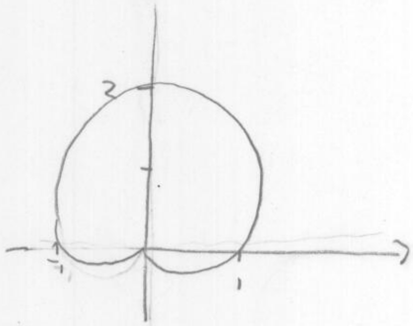
$$+ \left[\left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta\right]$$

$$= \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]$$

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example

Length of a cardioid?



$$\begin{cases} r = 1+\sin\theta \end{cases} \quad 0 \le \theta \le 2\pi \qquad \frac{dr}{d\theta} = \cos\theta$$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

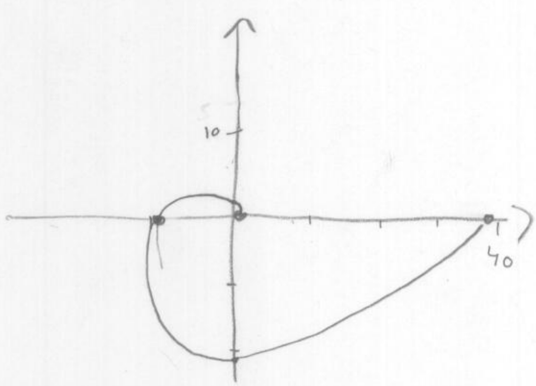
$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

$\big\downarrow$ computer algebra or messy integral

$L = 8$

Example

Length $r = \theta^2$, $0 \le \theta \le 2\pi$



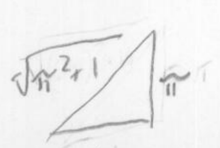
$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\theta^4 + (2\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad \theta = 2\tan t$$

$$d\theta = 4\sec^2\theta\, d\theta$$

$$= \int_0^{\arctan(\pi)} 4\tan t\,(2\sec t)(4\sec^2 t)\,dt$$



$$= 32\int_1^{\sqrt{\pi^2+1}} u^2\,du$$

$u = \sec t$

$du = \sec t \tan t\, dt$

$$= 32\left(\frac{1}{3}u^3\right)\Big|_1^{\sqrt{\pi^2+1}}$$

$$= \frac{32}{3}\left[(\pi^2+1)^{3/2} - 1\right]$$

7