d Question: Area Under Parametric Curves

$$(x, y) = (g(t), f(t)) \quad \text{for} \quad \alpha \leq t \leq \beta$$

Area under $y = F(x)$ is $\int_a^b F(x)\,dx$ if $F(x) \geq 0$. But this is the same as insisting that $A = \int_a^b y\,dx$

$$= \int_\alpha^\beta f(t)\, g'(t)\,dt$$

Make the substitution
$y = f(t)$
$dx = g'(t)\,dt$.

Example (Area under an arch of the cycloid)

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$

$$A = \int_0^{2\pi r} y\,dx$$

$$= \int_0^{2\pi} r(1-\cos\theta)\left[r(1-\cos\theta)\,d\theta\right]$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} 1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta))\,d\theta$$

$$= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

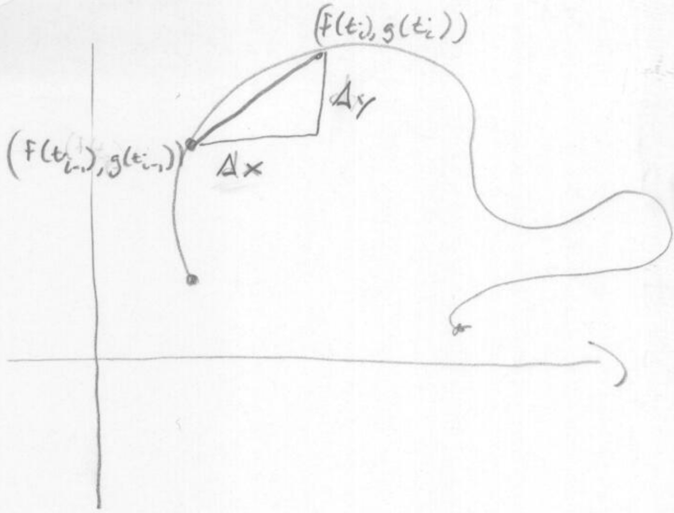
$$= r^2 \left[\left(\tfrac{3}{2}(2\pi) - 0 + 0\right) - (0 - 0 + 0)\right]$$

$$= \boxed{3\pi r^2}$$

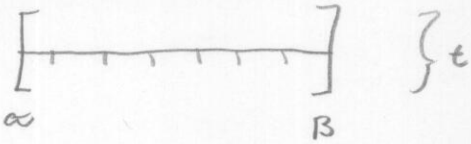
Example: $\begin{cases} x = t \\ y = f(t) \end{cases}$ $\int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \frac{dy}{dt} / \frac{dx}{dt}} \frac{dx}{dt}\, dt = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \frac{dy}{(dt)^2}}\, dt$



## exciting: Arc Length




Divide




into subintervals corresponding to little pieces of the curve

Length of curve from $(f(t_{i-1}), g(t_{i-1}))$ to $(f(t_i), g(t_i))$ is approximated by

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

However $\begin{cases} \Delta x \sim dx = f'(t_{i-1}) \Delta t \\ \Delta y \sim dy = g'(t_{i-1}) \Delta t \end{cases}$

$\Rightarrow$ Length of curve from $(f(t_{i-1}), g(t_{i-1}))$ to $(f(t_i), g(t_i))$ $\approx \sqrt{[f'(t_{i-1}) \Delta t]^2 + [g'(t_{i-1}) \Delta t]^2} = \sqrt{[f'(t_{i-1})]^2 + [g'(t_{i-1})]^2}\, \Delta t$

Total Length of curve $\approx \sum_{i=1}^{n} \sqrt{[f'(t_{i-1})]^2 + [g'(t_{i-1})]^2}\, \Delta t$

__Thm__ If a curve is described by $\alpha \le t \le b$, $(x, y) = (f(t), g(t))$ and $f', g'$ are cts, then

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

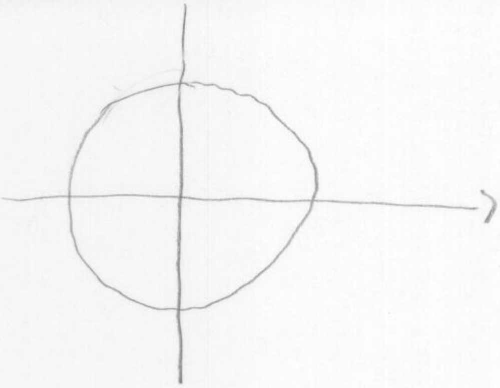
$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note if $\begin{cases} x = t \\ y = g(t) \end{cases}$ this reduces to the usual formula. $\left(\text{Or if } \begin{cases} x = f(t) \\ y = t \end{cases}\right)$.

<u>Example</u> Arclength of the circle (again)

Circle

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad 0 \le t \le 2\pi$$



$$\frac{dx}{dt} = -r\sin t \qquad \frac{dy}{dt} = r\cos t$$

$$L = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{r^2}\, dt$$

$$= \int_0^{2\pi} r\, dt$$

$$= 2\pi r$$

This fairly clearly illustrates that parametric expressions of a curve are easier sometime than Cartesian representatives.

<u>Example</u> Length of an arc of the cycloid?

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases} \qquad \frac{dx}{d\theta} = r(1-\cos\theta) \qquad \frac{dy}{d\theta} = r\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{r^2\left((1-\cos\theta)^2 + \sin^2\theta\right)}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \qquad \frac{1}{2}(1-\cos\theta) = \sin^2\left(\frac{\theta}{2}\right)$$

$$= \int_0^{2\pi} r\sqrt{2 - 2\cos\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 2r\int_0^{2\pi} \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

on $0 \le \theta \le 2\pi$, $\sin\left(\frac{\theta}{2}\right) > 0$

$$= 2\pi\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 4\pi\left(-\cos(\theta/2)\right)\Big|_0^{2\pi}$$

$$= 4\pi(-(1) - (-1)) = 8r$$

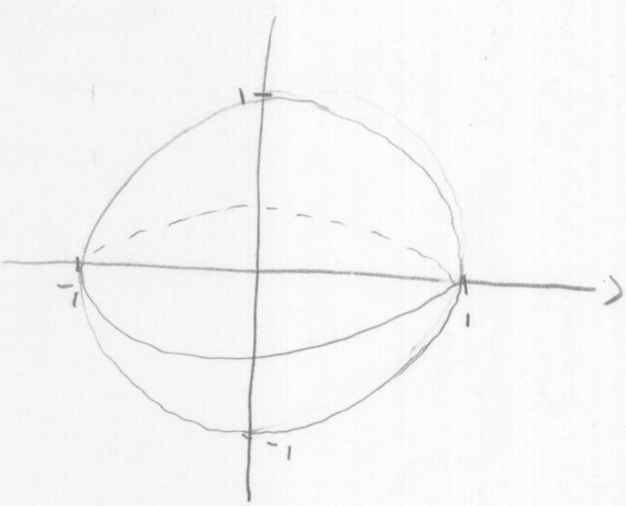
**Surface Area** Once we have a formula for $ds$, we can easily reconstruct surface area.

$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \end{cases}$$

About x-axis
$$SA = \int 2\pi y \, ds$$
$$= \int 2\pi g(t) \sqrt{f'(t)^2 + g'(t)^2} \, dt$$
$$= \int 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

About y-axis
$$SA = \int 2\pi x \, ds$$
$$= \int 2\pi f(t) \sqrt{f'(t)^2 + g'(t)^2} \, dt$$
$$= \int 2\pi x \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \, dt$$

**Example** Surface area of a sphere




$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad 0 \le t \le \pi$$

only need upper semicircle

- $ds = r\,dt$ from previous example
- Rotate around x-axis

$$SA = \int_0^{\pi} 2\pi y \cdot ds$$
$$= 2\pi \int_0^{\pi} r\sin t \, (r\,dt)$$
$$= 2\pi r^2 \int_0^{\pi} \sin t \, dt$$
$$= 2\pi r^2 (-\cos t) \Big|_0^{\pi}$$
$$= 2\pi r^2 [-1-1]$$
$$= 4\pi r^2$$

Notice that this is the derivative of volume! We saw this relationship when we studied related rates.

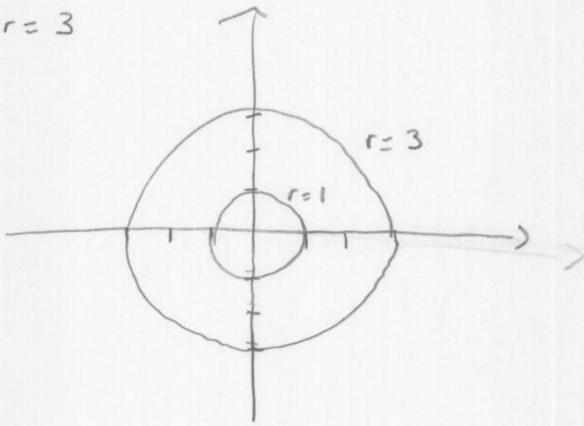
# Polar Curves

$r = f(\theta)$ $\qquad$ $F(r, \theta) = 0$

Graph consists of all points that have at least one representative $(r, \theta)$.

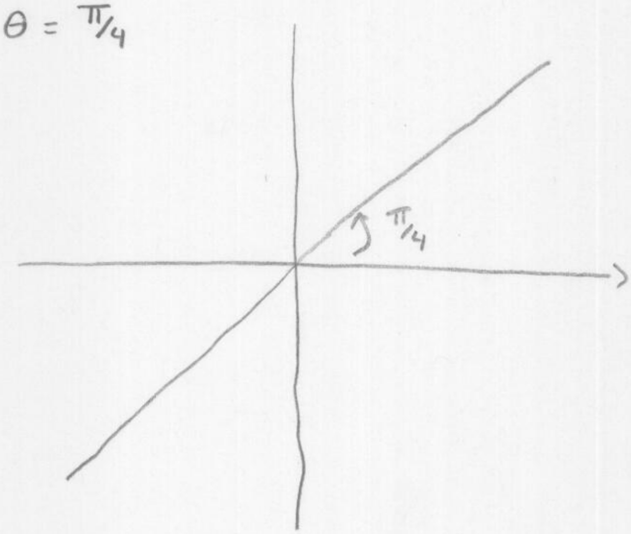
## Examples

① $r = 3$



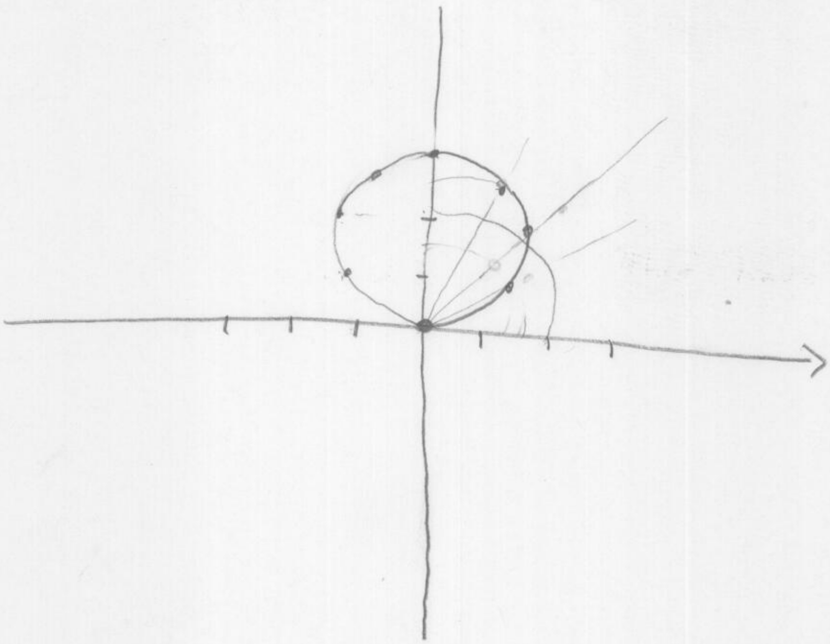

All points $(3, \theta)$

② $\theta = \pi/4$




All point $(r, \frac{\pi}{4})$

(3) $r = 3\sin\theta$ - Plot curve and find a Cartesian equation.



| $\theta$ | $r$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{6}$ | $\frac{3}{2}$ |
| $\frac{\pi}{4}$ | $\frac{3}{\sqrt{2}}$ |
| $\frac{\pi}{3}$ | $\frac{3\sqrt{3}}{2}$ |

**Cartesian Equation**

$r = 3\sin\theta$

$\sqrt{x^2+y^2} = 3\left(\frac{y}{r}\right)$

$\sqrt{x^2+y^2} = \frac{3y}{\sqrt{x^2+y^2}}$

$x^2+y^2 = 3y$

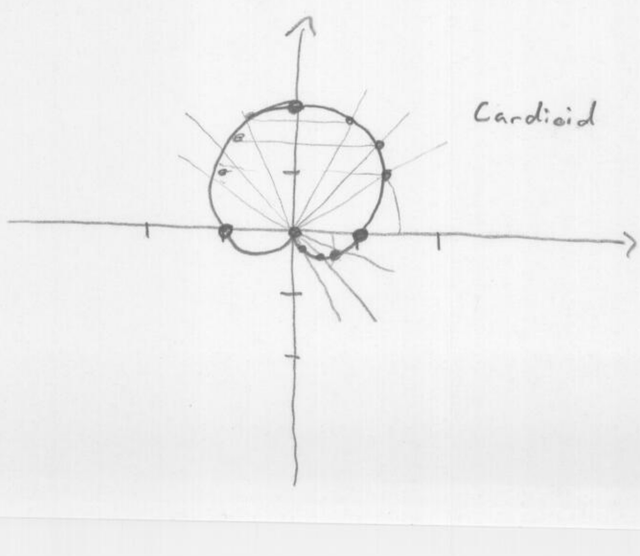
$\left(x^2+3y+\frac{9}{4}\right)+y^2 = 9/4$

$\left(x+\frac{3}{2}\right)^2+y^2 = \frac{9}{4}$ Another circle of radius $\frac{3}{2}$ centered at $\frac{3}{2}$.

$\sqrt{2} \sim 1.4$

$\sqrt{3} \sim 1.7$

**Example (4)** $r = 1+\sin\theta$

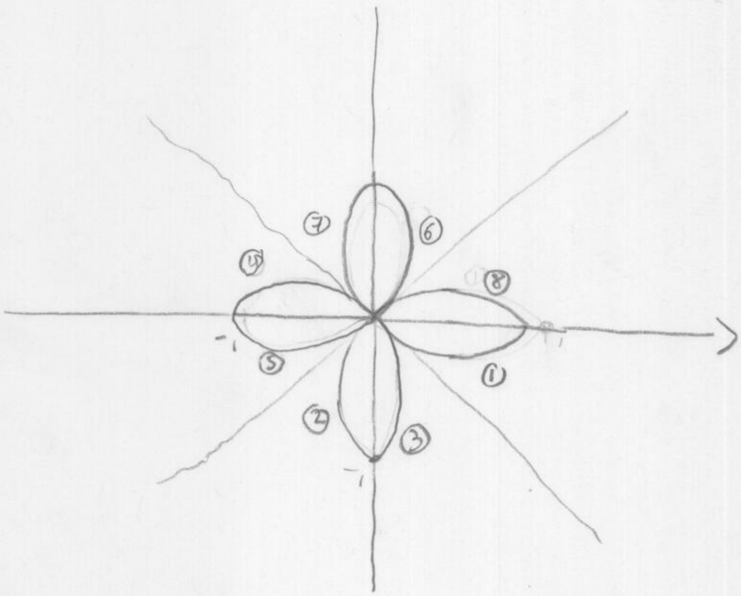


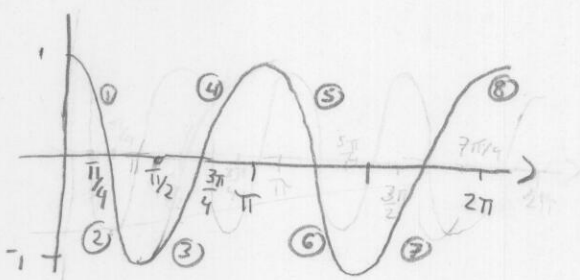
Cardioid

| $\theta$ | $r$ |
|---|---|
| $\frac{\pi}{6}$ | $3/2$ |
| $\frac{\pi}{4}$ | $1+\frac{\sqrt{2}}{2}$ |
| $\frac{\pi}{3}$ | $1+\frac{\sqrt{3}}{2}$ |

| | |
|---|---|
| $\frac{-\pi}{6}$ | $\frac{-1}{2}$ |
| $\frac{-\pi}{4}$ | $1-\frac{\sqrt{2}}{2}$ |
| $\frac{-\pi}{3}$ | $1-\frac{\sqrt{3}}{2}$ |

Example 5 $r=\cos(2\theta)$






Symmetry $F(r,\theta)=0$

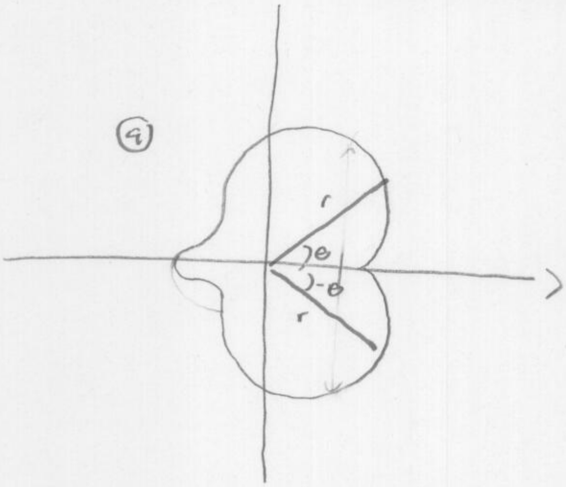
(a) If $F(r,\theta) = F(r,-\theta)$, curve is symmetric about polar axis. eg, $r=\cos 2\theta$

(b) If $F(r,\theta) = F(-r,\theta)$, curve is symmetric about origin

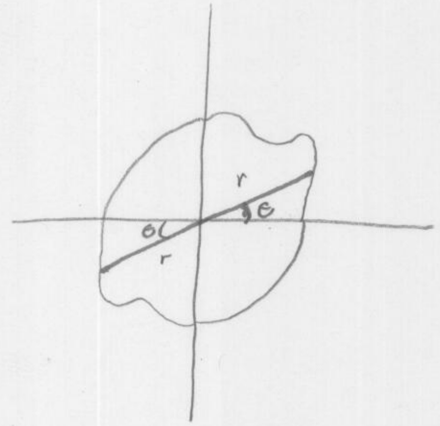
$F(r,\theta) = F(\cancel{r}\,\theta+\pi)$

(c) If $F(r,\theta) = F(r,\pi-\theta)$, curve is symmetric about $\theta=\frac{\pi}{2}$

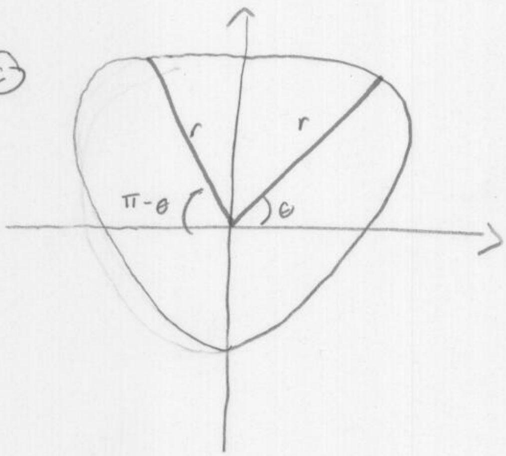
(a)




(b)




(c)




## Tangents

Suppose $r = f(\theta)$

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= f(\theta)\cos\theta \qquad = f(\theta)\sin\theta$$

] Both functions of a parameter $\theta$

Use what we learned about general parametric equations.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$