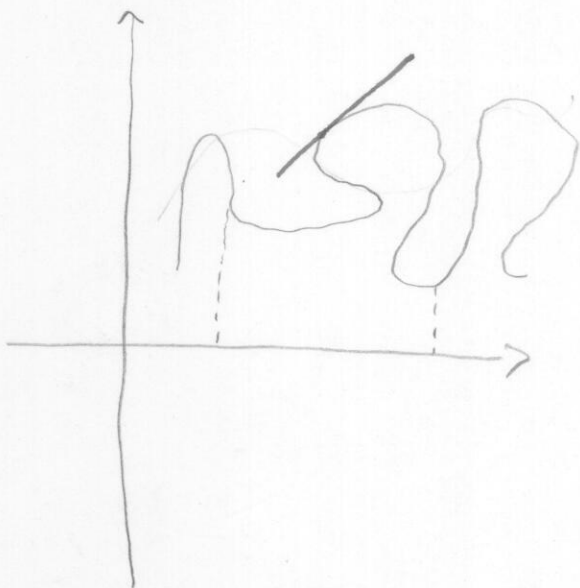


Curves defined by Parametric Equations

We might reasonably be interested in a curve which isn't a function of x or y .



- Still interested in slopes of tangent lines
- Still interested in area demarcated by curves

Think of a particle (or ant etc.) travelling along the curve above. At a given time t it has some x -coordinate and some y -coordinate.

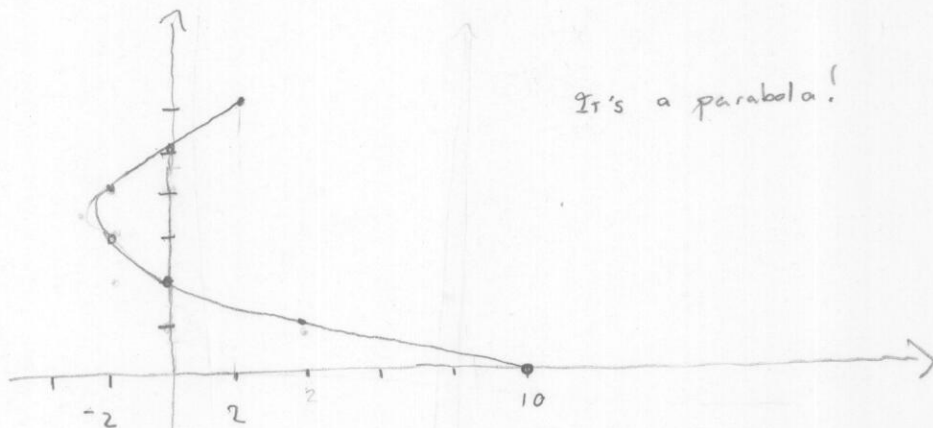
Position is $(x, y) = (f(t), g(t))$ two functions of t .

Example

$$x = t^2 - 3t$$

$$y = t + 2$$

	t	x	y
-2	-2	10	0
-1	-1	4	1
0	0	0	2
1	1	-2	3
2	2	-2	4
3	3	0	5
4	4	4	6



It's a parabola!

We can try to reduce to a cartesian equation.

$$x = t^2 - 3t$$

$y = t + 2$ ← We can solve for t , so this is doable.

$$t = y - 2$$

$$x = (y - 2)^2 - 3(y - 2)$$

$$x = y^2 - 4y + 4 - 3y + 6$$

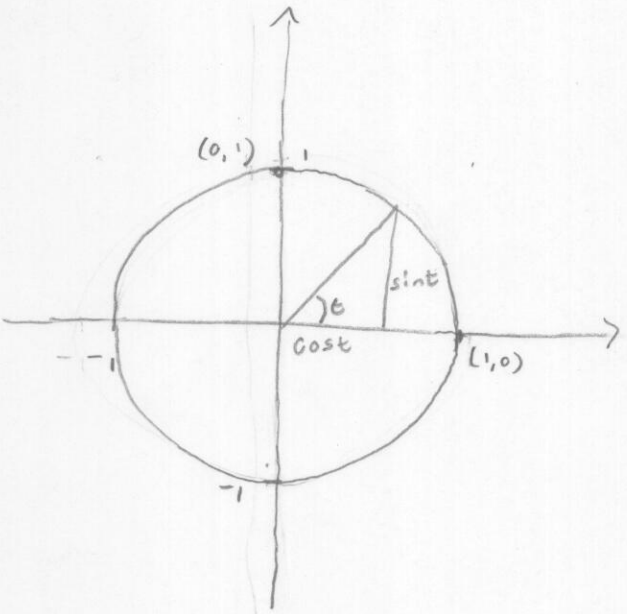
$$x = y^2 - 7y + 10$$

But this won't always be possible. Sometimes, also, we bound t , $a \leq t \leq b$,

so we get the arc of the curve running from $(f(a), g(a))$ to $(f(b), g(b))$.

Example

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \leq t \leq 2\pi$$



$t = a \rightarrow (x, y)$

t	x	y
0	1	0
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

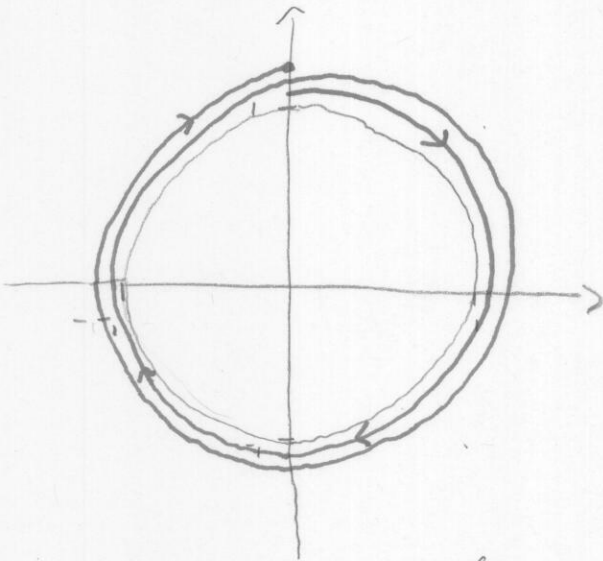
$$\cos^2 t + \sin^2 t = 1$$

$$x^2 + y^2 = 1$$

But what about

$$\begin{cases} x = \sin(2t) \\ y = \cos(2t) \end{cases} \quad 0 \leq t \leq 2\pi$$

Still gives points on the circle.



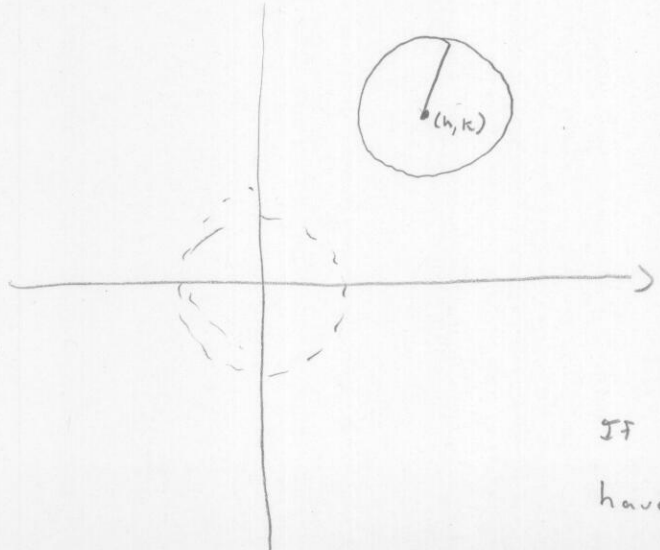
t	x	y
0	0	1
$\frac{\pi}{4}$	-1	0
$\frac{\pi}{2}$	0	-1
$\frac{3\pi}{4}$	-1	0

Go around the circle twice clockwise.

So different parametric curves (ways of travelling a path) can represent the same curve (set of points) in the plane.

Example

Equations for circle of center (h, k), radius r?



$$\begin{cases} x = r \cos t + h \\ y = r \sin t + k \end{cases} \quad 0 \leq t \leq 2\pi$$

If it was just at the origin, we'd have $\begin{cases} x = r \cos t \\ y = r \sin t \end{cases}$. We translate that curve to center at (h, k).

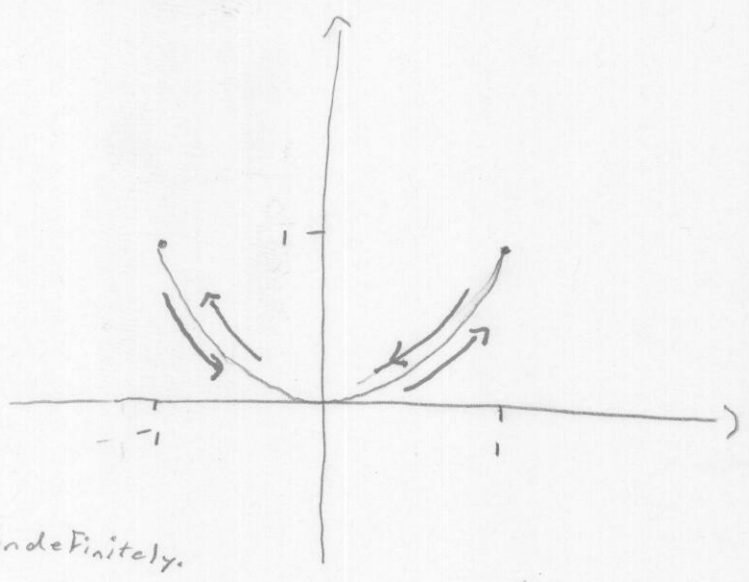
Example

$$\begin{cases} x = \cos t \\ y = \cos^2 t \end{cases}$$

$$y = (\cos t)^2 = x^2$$

But $-1 \leq \cos t \leq 1$

Vacillates back and forth indefinitely.



t	x	y
0	1	1
$\frac{\pi}{2}$	0	0
π	-1	1
$\frac{3\pi}{2}$	0	0
2π	1	1

A segue about graphing devices (handout).

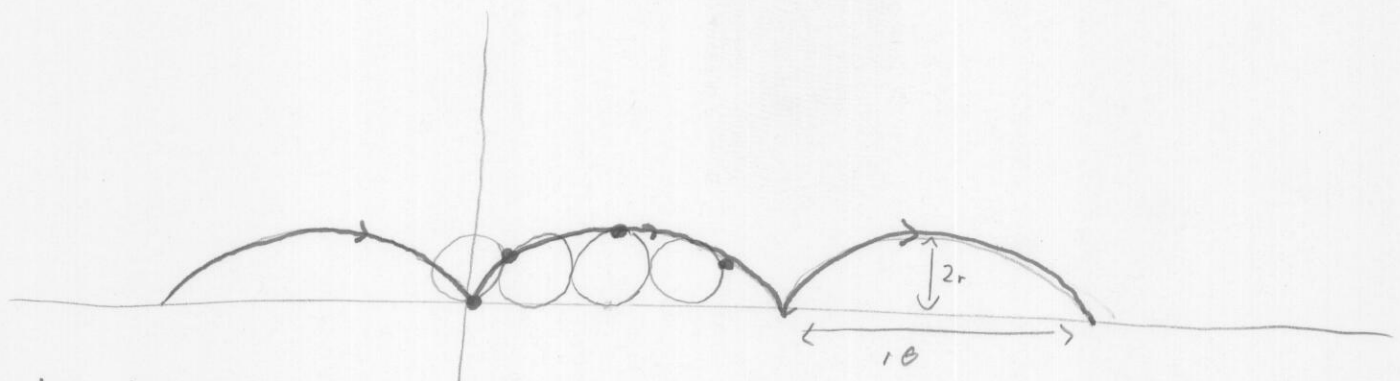
Graphing calculators have a parametric mode in which you can enter x, y in terms of functions of t . Curves are plotted as t increases, so this is often an illuminating way of understanding the graph. You can also draw shapes well beyond what you can plot by hand.

Note that you can always use $\begin{cases} x = t \\ y = g(t) \end{cases}$ to get a curve whose

Cartesian equation is $y = g(x)$, and similarly for $\begin{cases} x = f(t) \\ y = t \end{cases}$ and $x = f(y)$.

Parametric Curves as a way of tracing motion: The cycloid

Imagine a circle of radius r rolling in a straight line. We're interested in the path traced by a point on the circle.

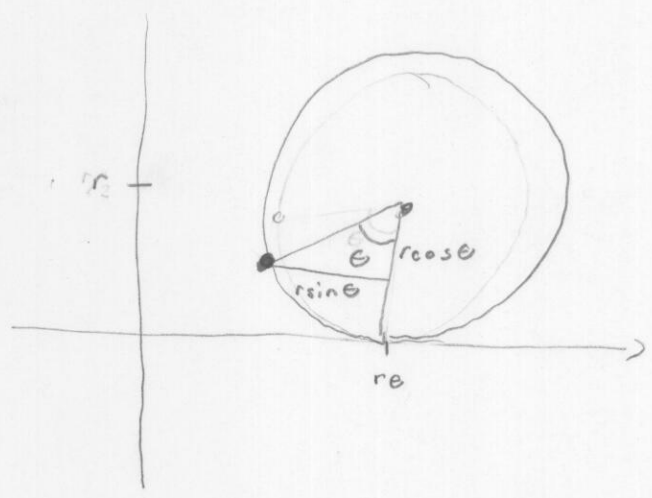


Long flat curve called a cycloid.

Let θ be the number of radians the circle has rotated ($\theta=0$ at $(0,0)$). After θ radians, where is center of circle?

$$\left(r\theta, r \right)$$

Distance rolled along axis
Height of center is always r



Our point is θ radians clockwise from the bottom of the circle (since the circle rolled counterclockwise, so by rolling it clockwise we could return the marked point to the bottom of the circle)

The vertical & horizontal differences from the center are $r \cos \theta$ and $r \sin \theta$.

Ergo

$$\begin{cases} x = r\theta - r \sin \theta = r(\theta - \sin \theta) \\ y = r - r \cos \theta = r\left(\frac{1}{2} - \cos \theta\right) \end{cases}$$

This is the "shortest" curve from a point A to a point B - particle acting under gravity takes least time to slide down it

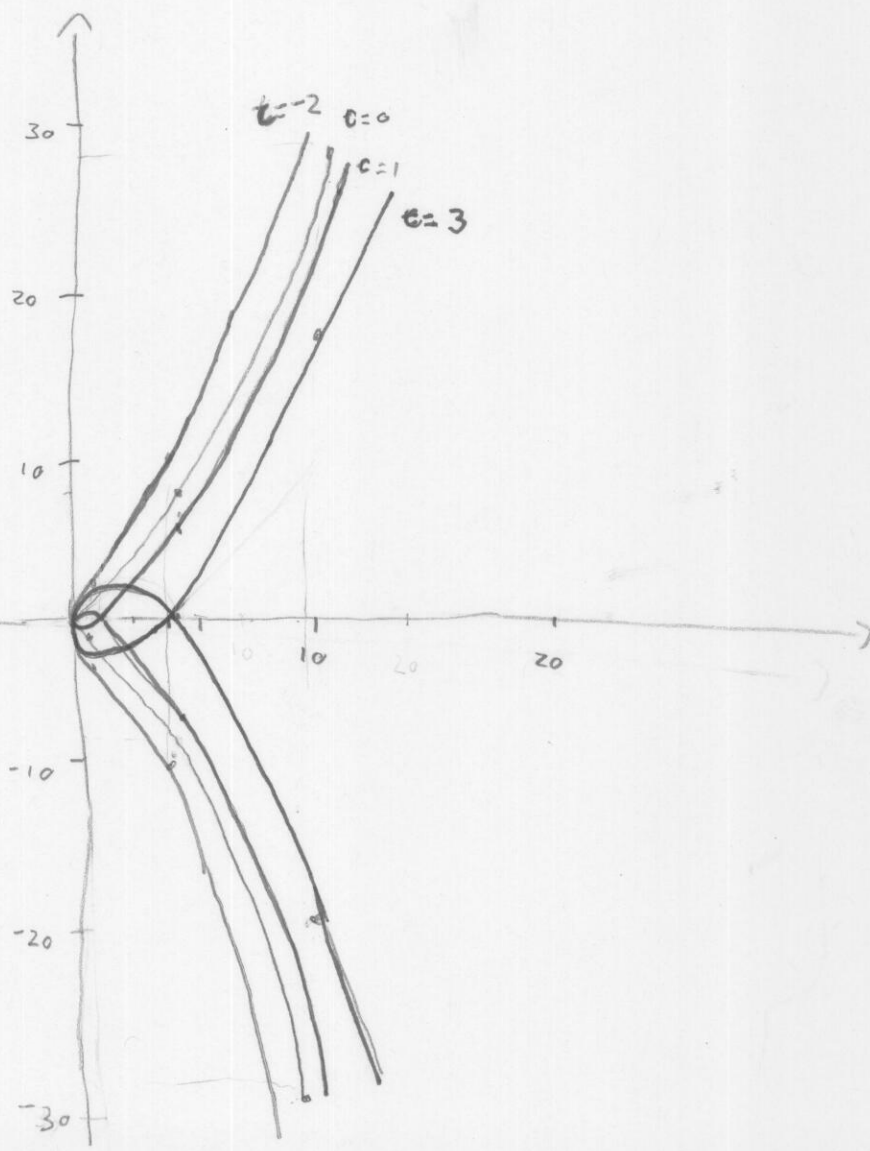
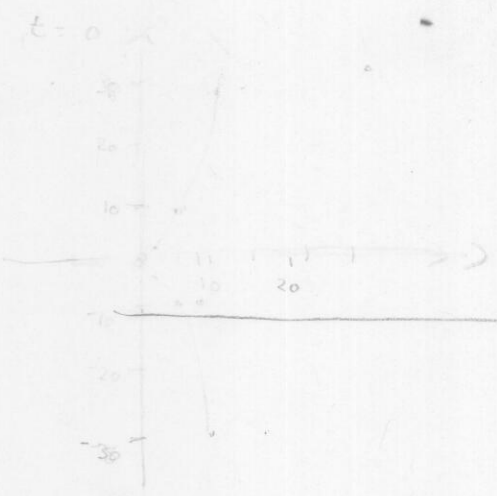
Families of parametric curves.

Changing a constant in a parametric curve tends to seriously change the shape of the curve.

$$\left\{ \begin{array}{l} x = a + c \cos t \\ y = a \tan t + c \sin t \end{array} \right. \quad \begin{array}{l} a = 0 \text{ Circle} \\ a << 0 \text{ or } a >> 0 \text{ two nearly straight lines} \end{array}$$

(See handout) or text pg. 641

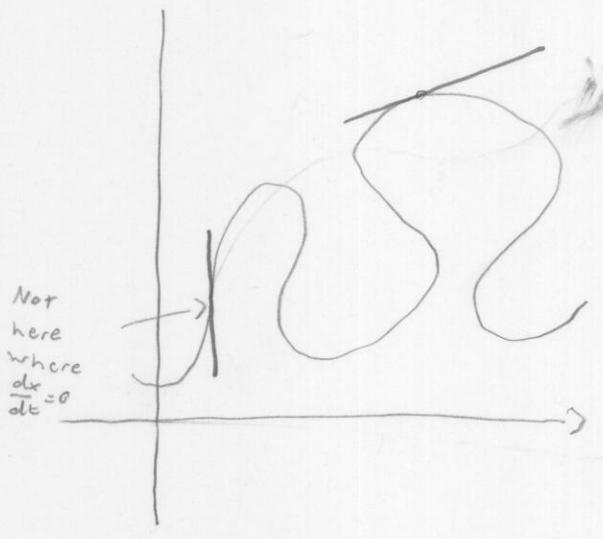
$$\left\{ \begin{array}{l} x = t^2 \\ y = t^3 - ct \end{array} \right. \quad \begin{array}{l} \text{Note always} \\ \text{Symmetric} \\ \text{about } x\text{-axis} \end{array}$$



Calculus w/ parametric curves

Two Questions : Slope of a Tangent Line / Area Under a Curve

Tangent Lines



$(x, y) = (g(t), f(t))$

We know $\frac{dx}{dt}$, $\frac{dy}{dt}$

We want $\frac{dy}{dx}$

Chain Rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

whenever $\frac{dx}{dt} \neq 0$

(that is, when the curve doesn't have a vertical tangent)

Rewritten

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)} \text{ if } g'(x) \neq 0$$

Similarly for higher derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

↑
replace y w/ $\frac{dy}{dx}$

Example

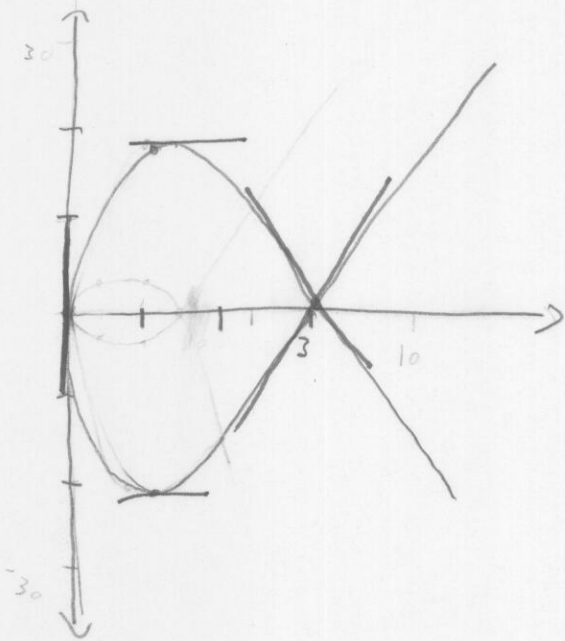
$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$$

In depth sketch of the loop region

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$



Some things to note

At (3, 0), two tangents

$$t = -\sqrt{3} \quad \frac{dy}{dx} = \frac{6}{2\sqrt{3}} = \sqrt{3}$$

$$t = \sqrt{3} \quad \frac{dy}{dx} = \frac{6}{-2\sqrt{3}} = -\sqrt{3}$$

Tangent horizontal

$$\frac{dy}{dt} = 0$$

$$3t^2 - 3 = 0$$

$$3t^2 = 3$$

$$t^2 = 1$$

$$t = \pm 1 \quad (1, 2) \quad (1, -2)$$

Tangent vertical

$$\frac{dx}{dt} = 0$$

$$2t = 0$$

$$t = 0 \quad (0, 0)$$

Concavity?

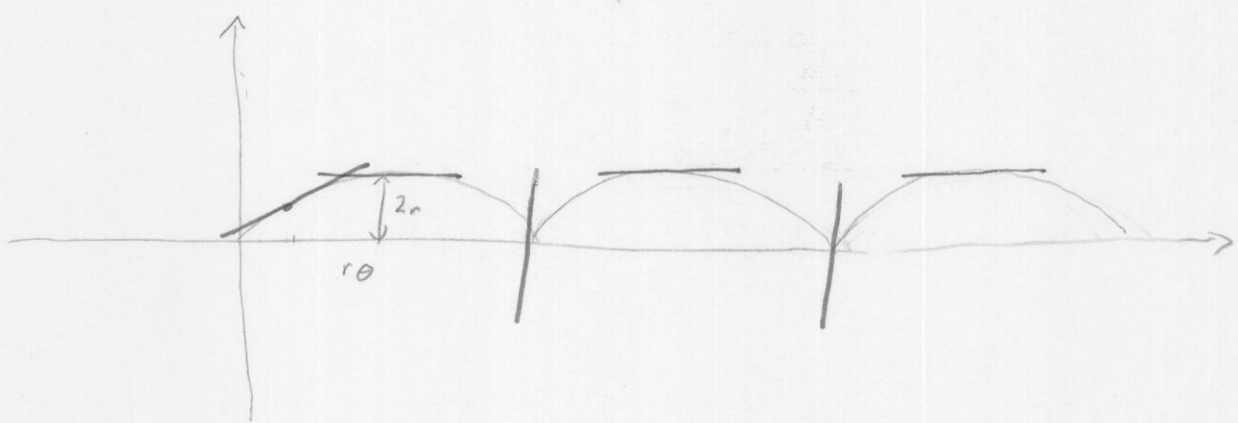
$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{1}{2t} \left(\frac{6t(2t) - 2(3t^2 - 3)}{(2t)^2} \right) = \frac{6(3t^2 + 3)}{4t^3} \\ &= \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{3(t^2 + 1)}{4(t^3)} \end{aligned}$$

Concave up when $t > 0$

Concave down when $t < 0$

Example 2

When does our cycloid have a vertical or horizontal tangent?



$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$

$$\frac{dx}{d\theta} = r(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = r(\sin\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\sin\theta}{r(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

E.g. at $\theta = \frac{\pi}{3}$ $(r(\frac{\pi}{3} - \frac{\sqrt{3}}{2}), \frac{r}{2})$ $\frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \sqrt{3}$

Horizontal $\frac{dy}{d\theta} = 0$ but $\frac{dx}{d\theta} \neq 0$

$$r\sin\theta = 0 \quad r(1 - \cos\theta) \neq 0$$

$$\sin\theta = 0 \quad \cos\theta \neq -1$$

$$\theta = \pi, 3\pi, 5\pi, \dots$$

Vertical

$$\lim_{\theta \rightarrow \theta_0} \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \pm\infty \quad \text{At } \theta = 2n\pi, \quad 1 - \cos\theta \text{ and } \sin\theta \text{ are both } 0.$$

$$\lim_{\theta \rightarrow 2n\pi^+} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta \rightarrow 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty \quad \text{and} \quad \lim_{\theta \rightarrow 2n\pi^-} \frac{\sin\theta}{1 - \cos\theta} = -\infty$$

Second Question: Area

Stick $(x, y) = (g(t), f(t))$ for $\alpha \leq t \leq \beta$

Area under $y = F(x)$ is $\int_a^b F(x) dx$ if $F(x) \geq 0$. But this is the

same as insisting that $A = \int_a^b y dx$

Make the substitution

$$= \int_{\alpha}^{\beta} f(t) g'(t) dt$$

$$y = f(t) \\ dx = g'(t) dt.$$

Example (Area under an arch of the cycloid)

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$

$$A = \int_0^{2\pi r} y dx$$

$$= \int_0^{2\pi} r(1 - \cos\theta) [r(1 - \cos\theta) d\theta]$$

$$= r^2 \int_0^{2\pi} (1 - \cos\theta)^2 d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)) d\theta$$

$$= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin 2\theta \right]_0^{2\pi}$$

$$= r^2 \left[\left(\frac{3}{2}(2\pi) - 0 + 0 \right) - (0 - 0 + 0) \right]$$

$$= \boxed{3\pi r^2}$$