Improper Integrals (Section 7.8)

So far $\int_a^b f(x)\,dx$, where F is continuous on $[a,b]$

What about

- $[a,\infty)$?
- $(-\infty,b]$?
- F has a single discontinuity on $[a,b]$ e.g. $\frac{1}{x-2}$ on $[2,3]$

Goal

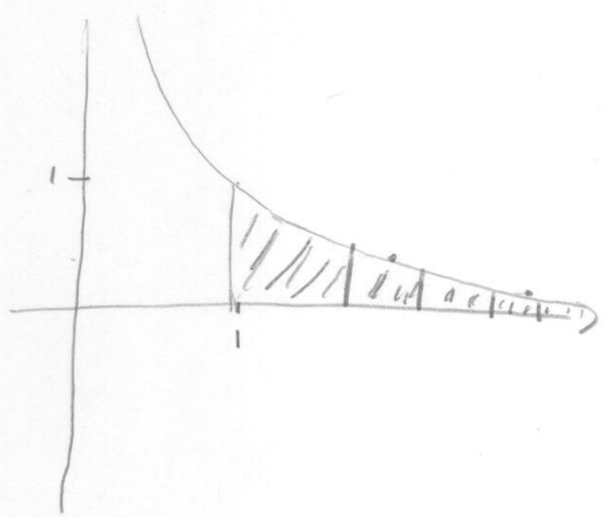
Use our computations of integrals over $[a,b]$ w/ f cts to approximate these "improper" integrals."

Example 1

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Can find for $t<\infty$.

$$\int_1^t \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_1^t = \frac{-1}{t} + \frac{-(-1)}{1} = 1 - \frac{1}{t}$$



Let

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

Defn (An infinite improper integral, or improper integral of Type I)

(a) If $\int_a^t f(x)\,dx$ exists $\forall\ t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

(b) If $\int_t^b f(x)\,dx$ exists $\forall\ t \leq b$, then

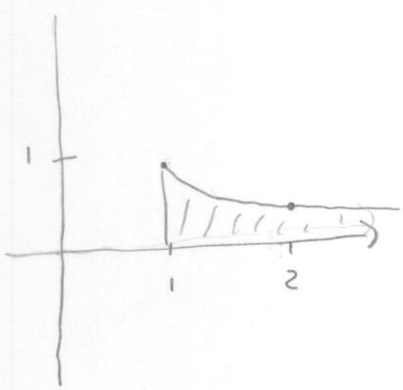
$$\int_{-\infty}^b f(x)\,dx = \lim_{t \to \infty} \int_t^b f(x)\,dx$$

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are convergent if the limit exists and divergent if not.

(c) $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \quad \text{for any } a \in \mathbb{R}.$$

Example 2

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx = \lim_{t \to \infty} \ln|x| \Big|_1^t$$



$$= \lim_{t \to \infty} \ln|t| - \ln|1|$$

$$= \infty$$

diverges.

Example 3

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

Rapid Integration by Parts

$$\begin{array}{cc} x & e^x \\ 1 & e^x \\ 0 & e^x \end{array}$$

$$= \lim_{t\to-\infty}\left[xe^x - e^x\right]_t^0$$

$$= \lim_{t\to-\infty}\left[(te^t - e^t) - (0-1)\right]$$

$$= \left[\lim_{t\to-\infty} te^t\right] - 0 + 1$$

$$= \lim_{t\to-\infty}\frac{t}{e^{-t}} - 1$$

$t \to -\infty$

$e^{-t} \to \infty$

So we can use L'hospital's rule

$$= \lim_{t\to-\infty}\frac{1}{-e^{-t}} - 1$$

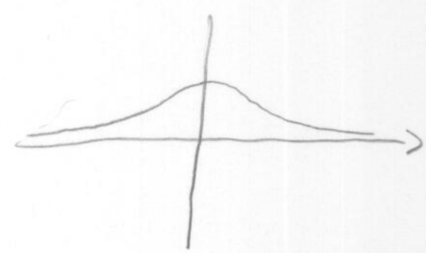
$$= 0 - 1$$

$$= -1$$

Converges

Example 4

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$



$$= \lim_{t\to-\infty}\int_t^0\frac{1}{1+x^2}\,dx + \lim_{s\to\infty}\int_0^s\frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\tan^{-1}x\Big|_t^0 + \lim_{s\to\infty}\tan^{-1}x\Big|_0^s$$

$$= \lim_{t\to\infty}\tan^{-1}0 - \tan^{-1}t + \lim_{s\to\infty}\tan^{-1}s - \tan^{-1}0$$

$$= 0 - \frac{-\pi}{2} + \frac{\pi}{2} - 0$$

$$= \pi$$

Example 5

For what values of p is $\int_1^\infty \frac{dx}{x^p}$ convergent?

- $p=1$ divergent
- $p \neq 1$ $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty} \frac{1}{(p-1)x^{p-1}}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{p-1}\left[\frac{1}{t^{p-1}} - 1\right]$$

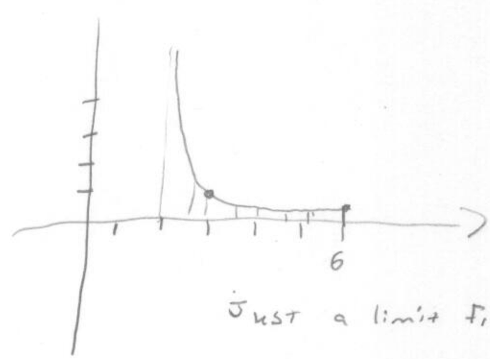
If $p>1$, goes to 0

If $p<1$, goes to ∞

So $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p \leq 1$.

What about discontinuities?

$F(x) = \frac{1}{\sqrt{x-2}}$ on $[2, 6]$, e.g.



Just a limit from the right.

Defn (Type II: Discontinuous Integrals)

(a) If f is cts on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx \quad \text{if the limit exists.}$$

(b) If f is cts on $(a, b]$, dis cts at a,

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \quad \text{if limit exists}$$

- Convergent if limits exist
- Divergent otherwise.

(c) If f is cts on $[a, c)$ and $(c, b]$ but not at c,

then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example 1

$$\int_2^6 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_t^6 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big|_t^6$$

$$= \lim_{t \to 2^+} \left[2\sqrt{4} - 2\sqrt{t-2}\right]$$

$$= 4 - 0$$

$$= 4$$

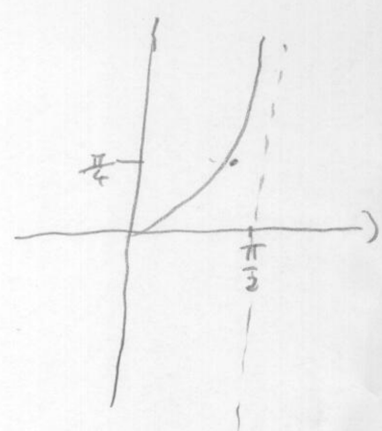
Example 2

$$\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{t \to \frac{\pi}{2}^-} \int_a^t \tan\theta\,d\theta$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln|\sec\theta|\,\Big|_0^t$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln|t| - \ln|0|$$

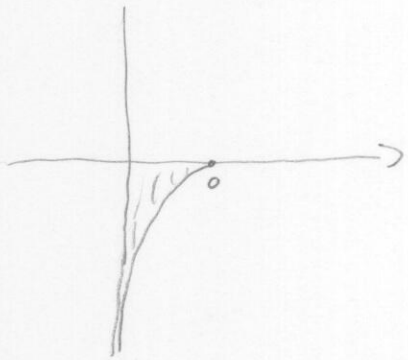
$$= \infty \quad \text{diverges!}$$



Example $\displaystyle\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$

$\ln x \quad dx$

$\frac{1}{x}dx \quad x$



$$= \lim_{t\to 0^+} \left[x\ln x \Big|_t^1 - \int_t^1 dx \right]$$

$$= \lim_{t\to 0^+} \left[\ln 1 - t\ln t - (1-t) \right]$$

$$= \lim_{t\to 0^+} \left[-t\ln t - 1 \right]$$

$$= \lim_{t\to 0^+} \frac{-\ln t}{1/t} - 1$$

$\ln t \to -\infty$ ✓

$\frac{1}{t} \to \infty$ ✓

Use L'Hospital

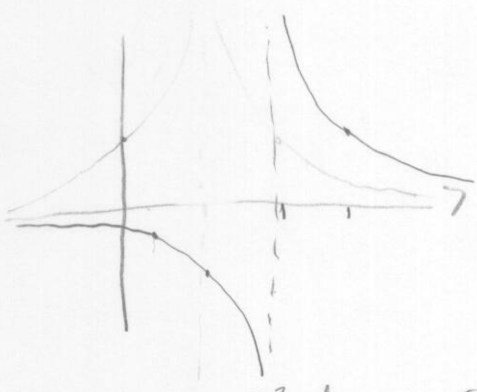
$$= \lim_{t\to 0^+} \frac{\frac{-1}{t}}{\frac{-1}{t^2}} - 1$$

$$= \lim_{t\to 0^+} t - 1$$

$$= \boxed{-1}$$

Example

$$\int_0^3 \frac{dx}{x-2}$$



Warning!

Not $\displaystyle\int_0^3 \frac{dx}{x-2} = \ln|x-2|\Big|_0^3$

$= \ln 1 - \ln|-2|$

$= \ln 2$

Must compute improper integrals in terms of limits.

$$\int_0^3 \frac{dx}{x-2} = \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2}$$

$$= \lim_{t\to 2^-} \int_0^t \frac{dx}{x-2} + \lim_{s\to 2^+} \int_s^3 \frac{dx}{x-2}$$

$$= \lim_{t \to 2^-} \ln|x-2| \Big|_0^t + \lim_{s \to 2^+} \ln|x-2| \Big|_s^3$$

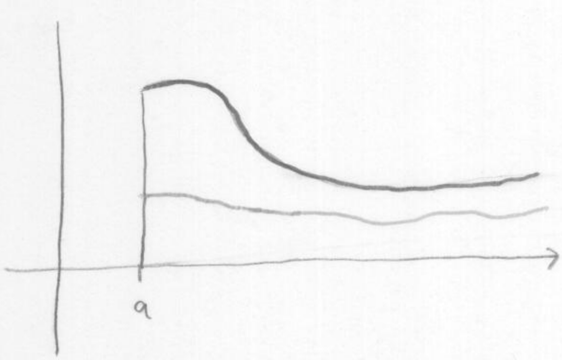
$$= \lim_{t \to 2^-} \underbrace{\ln|t-2| - \ln 0}_{-\infty} + \lim_{s \to 2^+} \underbrace{\ln|2| - \ln|s-2|}_{-\infty}$$

diverges

A note about symmetry: notice that even though finite areas here theoretetically cancel, integral still diverges. Likewise if f is odd, $\int_{-\infty}^{\infty} f(x)\,dx$ needn't be 0.

The comparison Test

Sometimes it's useful just to know if an integral converges or not



Suppose f and g are cts and

$f(x) \geq g(x) \geq 0$ on (a, ∞).

(a) If $\int_a^{\infty} f(x)\,dx$ converges, so does $\int_a^{\infty} g(x)\,dx$.

(b) If $\int_a^{\infty} g(x)\,dx$ diverges, so does $\int_a^{\infty} f(x)\,dx$)

Similarly for other improper integrals of positive funcions.

Converse is not usually true.

<u>Example 1</u>

$\int_0^\infty e^{-x^2}\,dx$ Can't evaluate directly

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

For $x > 1$ $\quad x^2 \geq x$

$$-x^2 \leq -x$$

$$0 \leq e^{-x^2} \leq e^{-x}$$

$$\int_1^\infty e^{-x}\,dx = \lim_{t \to \infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t \to \infty} -e^{-x}\Big)_1^t$$

$$= \lim_{t \to \infty} -e^{-t} - \frac{1}{e}$$

$$= \frac{1}{e} \quad \text{converges}$$

$\Rightarrow \int_1^\infty e^{-x^2}\,dx$ converges

$\Rightarrow \int_0^\infty e^{-x^2}\,dx \qquad =$

<u>Example 2</u> (can use this theorem to get rid of annoying terms)

$$\int_1^\infty \frac{2+e^{-x}}{x}\,dx \qquad \frac{2+e^{-x}}{x} > \frac{2}{x}$$

and $\int_1^\infty \frac{2}{x}\,dx$ diverges.