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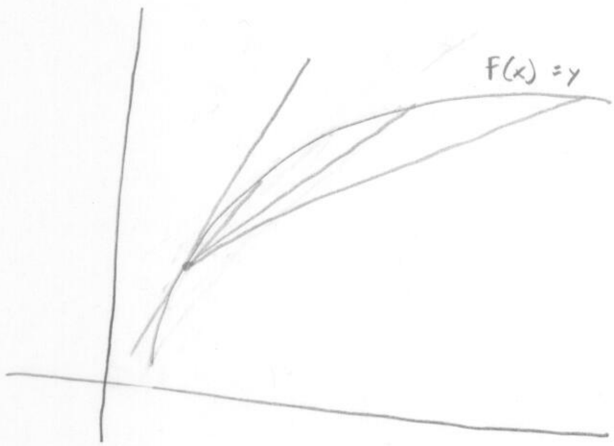
antidifferentiation formulas

(Preliminaries and Policies)

Two Questions In Calculus

(I) Rate of Change - What is the slope of a tangent line to a curve?



We approximated using slopes of secants,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Example If the position of a car is given by $p(x) = 3\sin x + \ln x + x^3 + 6$ what is its velocity as a function of x?

$$v(x) = p'(x) = 3\cos x + \frac{1}{x} + 3x^2$$

Important Rules

$$\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x)$$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$$

Chain Rule $$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\,g'(x)$$

Product Rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

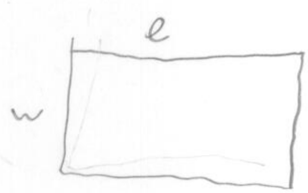
Quotient Rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ when $g(x) \neq 0$

Most Important Application: Maximazation & Minimization

Thm A function $f(x)$ can only attain a local max or local min at a point where $f'(x) = 0$ or does not exist.

(Mention existence of absolute maxima & minima)

Example Suppose a rectangle has perimeter 20. What should its length and width be to maximize its area?



$A = lw$

$P = 2l + 2w$

$20 = 2l + 2w$

$10 = l + w$

$10 - l = w$

$A = l(10 - l)$

$A(l) = 10l - l^2$

$A'(l) = 10 - 2l$

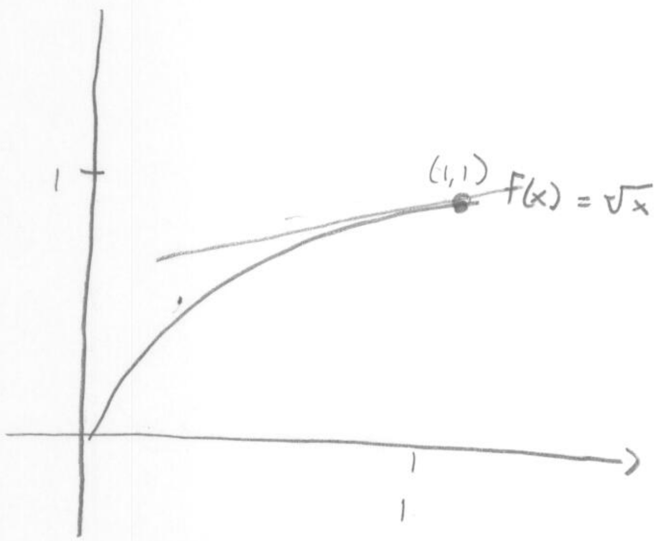
$0 = 10 - 2l$

$$\boxed{\begin{aligned} 5 &= l \\ 5 &= w \end{aligned}}$$

(Mention

Application: Linear Approximation

Tangent line approximates a function locally.



$\sqrt{1.2} = ?$

Slope tangent line

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(1) = \frac{1}{2}$

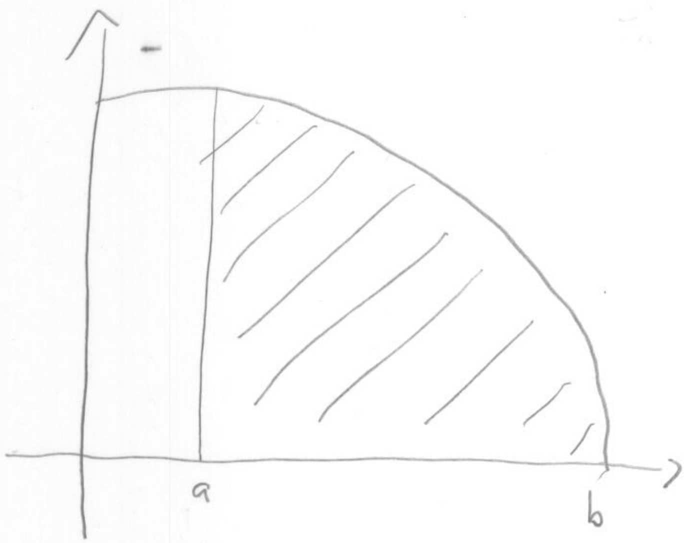
Tangent line at $x = 1$

$L(x) - 1 = \frac{1}{2}(x-1)$

$L(x) = \frac{1}{2}x + \frac{1}{2}$

$$\sqrt{1.2} = f(1.2) \approx L(1.2) = \tfrac{1}{2}(1.2) + \tfrac{1}{2}$$
$$= 1.1$$

II Total change - What is the area under a curve?



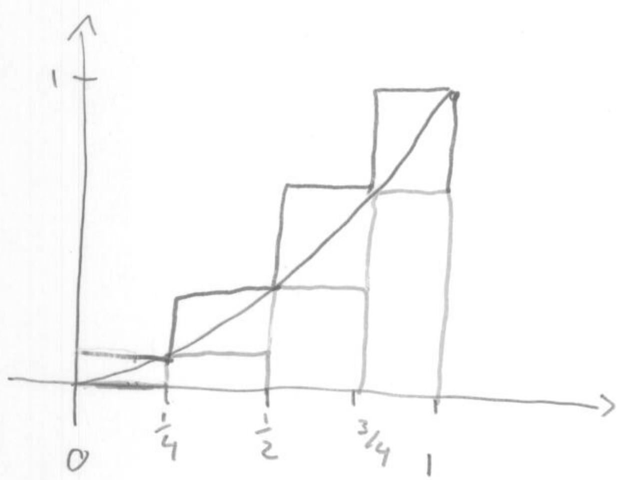
First, what do we mean by area?

We understand the area of a rectangle, so we approximate our region by rectangles.

(Explain why this is total change)

(Example of doing this by hand)

<u>Example</u> $f(x) = x^2$



A <u>Riemann sum</u> is an approximation of the area under a curve by a finite number of rectangles.

n = # rectangles

$\frac{b-a}{n}$ = length of interval

x_* = sample point in interval (often an endpoint for calculations)

$$R = \sum_{i=1}^{4} f(x_i^*)\,\Delta x$$

$$L = f(0)\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right)\tfrac{1}{4}$$

$$= \tfrac{1}{4}\left(f(0) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{2}{4}\right) + f\left(\tfrac{3}{4}\right)\right)$$

$$= \tfrac{1}{4}\left(0 + \tfrac{1}{16} + \tfrac{4}{16} + \tfrac{9}{16}\right) = \boxed{\tfrac{7}{32}}$$

$$R = f(\tfrac{1}{4})(\tfrac{1}{4}) + f(\tfrac{1}{2})(\tfrac{1}{4}) + f(\tfrac{3}{4})(\tfrac{1}{4}) + f(1)(\tfrac{1}{4})$$

$$= \left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}\right)\left(\frac{1}{4}\right)$$

$$= \boxed{\frac{15}{32}}$$

$$\frac{7}{32} \le A \le \frac{15}{32}$$

We could take a finer subdivision and get a better approximation. Say we take 8 subintervals.

$$L = \frac{1}{8}\left(f(0) + f(\tfrac{1}{8}) + \dots + f(\tfrac{7}{8})\right)$$

$$= \frac{1}{8}\left(\frac{140}{64}\right)$$

$$= \frac{35}{128}$$

$$R = \frac{1}{8}\left(f(\tfrac{1}{8}) + \dots + f(\tfrac{8}{8})\right)$$

$$= \frac{1}{8}\left(\frac{204}{64}\right)$$

$$= \frac{51}{128}$$

Defn (The Definite Integral) The area under a curve $f(x)$ from a to b is

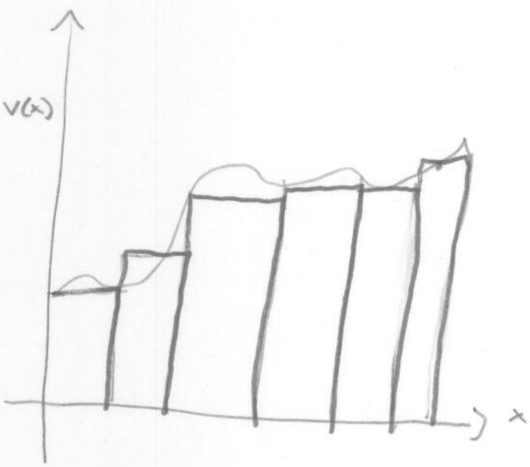
$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

This is a deplorably nasty definition to compute with.

The punchline of Calculus I is that these two questions are related.

Example

Integral of velocity $\approx$ Position



(Define antiderivatives, do an example)

Theorem (Fundamental Theorem of Calculus) If f is cts on $[a, b]$

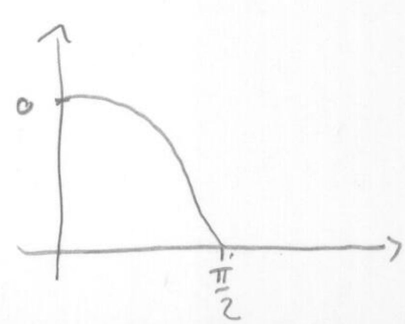
I. If $g(x) = \int_a^x f(t)\,dt$, $g'(x) = f(x)$, for $x \in (a, b)$

II. If $f(x) = F'(x)$, that is, if $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x) = F(b) - F(a)$$

Examples

$$\int_0^{\pi/2} \cos x = \sin x \Big|_0^{\pi/2}$$



$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1$$

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{3}(1) - \frac{1}{3}(0)$$

How was this proved?

Part I

We observe the following rules for integration:

① $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

④ $\int_a^a f(x) = 0$

② $\int_a^b c\,f(x)\,dx = c\int f(x)\,dx$

③ $g(x) \le f(x) \le h(x)$ on $[a,b] \Rightarrow \int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$

③ If $m \le f(x) \le M$ on $[a,b]$, $mh \le \int_a^b f(x)\,dx \le Mh$

Let $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \int_x^{x+h} f(t)\,dt$$

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

On $[x, x+h]$, f has a maximum $M(h) = f(u_h)$ & a minimum $m(h) = f(v_h)$

$$\frac{1}{h}h(m(h)) \le \frac{g(x+h)-g(x)}{h} \le \frac{1}{h}h(M(h))$$

But as $h \to 0$ $\quad m(h) = F(u_h) \to F(x)$

$u_h \to x$ $\quad \Rightarrow$

$v_h \to x$ $\quad M(h) = F(v_h) \to F(x)$

So $\displaystyle\lim_{h\to 0} m(h) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le \lim_{h\to 0} M(h)$

$$F(x) \le g'(x) \le F(x) \quad \Big\} \text{ Squeeze Thm}$$

Examples

$$g(x) = \int_a^x \frac{dt}{1+t^2} \qquad g'(x) = \frac{1}{1+x^2}$$

$$g(x) = \int_a^{\sqrt{x}} \sin x \, dx \qquad g'(x) = \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

Part II Want to Find $\int_a^b F(x)\,dx$

$g(x) = \displaystyle\int_a^x F(t)\,dt$. Want $g(b)$. Note $g(a) = 0$

$g'(x) = F(x)$

Any antiderivative of $F(x)$ is $F(x) = g(x) + c$.

$$\int_a^b F(t)\,dt = g(b)$$

$$= g(b) - g(a)$$

$$= F(b) - F(a)$$

Note $\int_a^b f(x)\,dx$ is a number