S1012.001 Summer 2011
Calculus II

Final

Instructions: You have 90 minutes to complete the exam. There are seven problems, worth a total of 120 points. (Plus some extra credit in case you get bored.) Calculators and textbooks are not allowed. Provide the answers in the simplest possible form that does not require calculator use. (E.g. expressions like $\sqrt{13}$ are fine.) Show all of your work: if you only give the answer you will receive no credit, but conversely, partial credit will be given for partial solutions.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	5	
2	20	
3	20	
4	25	
5	15	
6	15	
7	20	
8	0	
Total:	120	

Problem 1. *5pts.*

Evaluate the following integral.

$$\int \cos x \ln(\sin x) dx$$

Solution: First let $t = \sin(x)$, $dt = \cos x dx$, reducing the integral to $\int \ln(t)dt$. Then let $u = \ln(t)$, $dv = dt$, so that $du = \frac{dt}{t}$ and $v = t$. We see $\int \ln(t)dt = t\ln(t) - \int t\frac{dt}{t} = t\ln(t) - t + C$. Undoing the substitution we obtain $\int \cos x \ln(\sin x)dx = \sin x \ln(\sin x) - \sin x + C$.

Problem 2.

(a) [10pts.] Find the length of the curve $f(x) = \int_1^x \sqrt{\sqrt{t}-1}dt$ for $1 \leq x \leq 16$.

Solution: Notice that $f'(x) = \sqrt{\sqrt{x}-1}$ by the Fundamental Theorem of Calculus Part I. So the length of the curve is

$$\begin{aligned} L &= \int_1^{16} ds \\ &= \int_1^{16} \sqrt{1+[f'(x)]^2}dx \\ &= \int_1^{16} \sqrt{1+\sqrt{x}-1}dx \\ &= \int_1^{16} \sqrt{\sqrt{x}}dx \\ &= \int_1^{16} x^{\frac{1}{4}}dx \\ &= \frac{4}{5}\left[x^{\frac{5}{4}}\right]_1^{16} \\ &= \frac{4}{5}(32-1) \\ &= \frac{124}{5} \end{aligned}$$

(b) [10pts.] Find the surface area of the solid generated by rotating the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ on $1 \leq x \leq 2$ about the y-axis.

Solution: We see $\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x} = \frac{x^2-1}{2x}$. Ergo

$$\begin{aligned}
SA &= \int_1^2 2\pi x ds \\
&= 2\pi \int_1^2 x\sqrt{1 + \left[\frac{x^2-1}{2x}\right]^2} dx \\
&= 2\pi \int_1^2 x\sqrt{1 + \frac{x^4 - 2x^2 + 1}{4x^2}} dx \\
&= 2\pi \int_1^2 x\sqrt{\frac{x^4 + 2x^2 + 1}{4x^2}} dx \\
&= 2\pi \int_1^2 x\sqrt{\frac{(x^2+1)^2}{4x^2}} dx \\
&= 2\pi \int_1^2 x \frac{x^2+1}{2x} dx \\
&= \pi \int_1^2 \left[x^2 + 1\right] dx \\
&= \pi \left[\frac{x^3}{3} + x\right]_1^2 dx \\
&= \frac{10\pi}{3}
\end{aligned}$$

Problem 3.

Let $x = e^t \cos(t), y = e^t \sin(t)$, for $0 \leq t \leq \pi$.

(a) [10pts.] Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution:

We have $\frac{dx}{dt} = e^t \cos t + e^t(-\sin t)$ and $\frac{dy}{dt} = e^t \sin t + e^t \cos t$. Thus

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
&= \frac{e^t(\sin t + \cos t)}{e^t(\cos t - \sin t)} \\
&= \frac{\cos t + \sin t}{\cos t - \sin t}
\end{aligned}$$

Moreover

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\
&= \frac{\frac{(-\sin t+\cos t)(\cos t-\sin t)-(\cos t+\sin t)(-\sin t-\cos t)}{\cos t-\sin t)^2}}{e^t(\cos t-\sin t)} \\
&= \frac{(1-2\sin t\cos t)-(-1-2\cos t\sin t)}{e^t(\cos t-\sin t)} \\
&= \frac{2}{e^t(\cos t-\sin t)^3}
\end{aligned}
$$

(b) [10pts.] Find the exact length of the curve.

Solution:

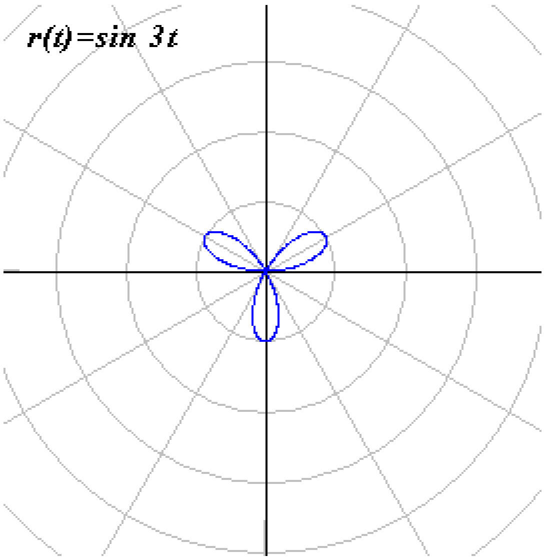
$$
\begin{aligned}
L &= \int ds \\
&= \int_0^{\pi} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt \\
&= \int_0^{\pi} \sqrt{[e^t(\cos t-\sin t)]^2 + [e^t(\cos t+\sin t)]^2}\,dt \\
&= \int_0^{\pi} e^t\sqrt{(1-2\cos t\sin t)+(1+2\cos t\sin t)}\,dt \\
&= \int_0^{\pi} e^t\sqrt{2}\,dt \\
&= \left[\sqrt{2}e^t\right]_0^{\pi} \\
&= \sqrt{2}[e^{\pi}-1]
\end{aligned}
$$

Problem 4.

(a) [5pts.] Sketch the curve $r = \sin(3\theta)$ for $0 \le \theta \le \pi$.

We obtain a three-leafed rose (see below).

(b) [7pts.] Find the tangent line to the curve at $(1, \frac{\pi}{6})$.



Solution:
Notice that $\frac{dr}{d\theta} = 3\cos(3\theta)$. Therefore we can compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$
$$= \frac{3\cos(3\theta)\sin\theta + \sin(3\theta)\cos\theta}{3\cos(3\theta)\cos\theta - \sin(3\theta)\sin(\theta)}$$

So at $(1, \frac{\pi}{6})$ we have

$$\frac{dy}{dx} = \frac{3(0)\left(\frac{1}{2}\right) + (1)\left(\frac{\sqrt{3}}{2}\right)}{3(0)\left(\frac{\sqrt{3}}{2}\right) - 1\left(\frac{1}{2}\right)}$$
$$= -\sqrt{3}$$

In Cartesian coordinates, the point in question is $\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right)$, or $\left(\frac{\sqrt{3}}{2}, \sqrt{12}\right)$. Hence the tangent line is $y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)$, or $y = -\sqrt{3}x + 2$.

(c) [8pts.] Find the area enclosed by a single loop of the curve.

Solution: A single loop of the curve runs from $\theta = 0$ to $\theta = \frac{\pi}{3}$. We compute the area:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} [\sin(3\theta)]^2 d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta) d\theta$$

Solution: Points of intersection can occur when $r = \frac{1}{2}$ *and* when $r = -\frac{1}{2}$, since the second equation also represents the circle of radius $\frac{1}{2}$. When $\frac{1}{2} = \sin(3\theta)$ and $0 \le \theta \le \pi$, the possibilities are $3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ and $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$. When $-\frac{1}{2} = \sin(3\theta)$ and $0 \le \theta \le \pi$, the possibilities are $3\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ and $\theta = \frac{7\pi}{18}, \frac{11\pi}{18}$.

Hence the six intersection points are $(\frac{1}{2}, \frac{\pi}{18})$, $(\frac{1}{2}, \frac{5\pi}{18})$, $(\frac{1}{2}, \frac{13\pi}{18})$, $(\frac{1}{2}, \frac{17\pi}{18})$, $(-\frac{1}{2}, \frac{7\pi}{18})$, $(-\frac{1}{2}, \frac{11\pi}{18})$. (Or any other representations of the same points.)

Problem 5.

(a) [5pts.] Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}}.$$

Solution:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n-1}} &= \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} + \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} \\
&= \frac{1}{1-\frac{1}{3}} + \frac{2}{1-\frac{2}{3}} \\
&= \frac{1}{\frac{2}{3}} + \frac{2}{\frac{1}{3}} \\
&= \frac{3}{2} + 6 \\
&= \frac{15}{2}
\end{aligned}$$

(b) [10pts.] For each of the series below, find some number n so that the nth partial sum of the series is within .001 of the actual sum of the series. It does not need to be the smallest such n. Justify your answer. (Do not attempt to compute this partial sum.)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Solution: The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ is alternating. Ergo the nth remainder R_n is less than the absolute value of the $(n+1)$st term, usually denoted b_{n+1}. Therefore we must find $n+1$ such that $R_n \leq \frac{1}{(n+1)!} \leq .001$. Letting $n+1 = 7$, so that $(n+1)! = 5040$, will do. Ergo the sixth partial sum of the series is accurate to within .001.

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ has $a_n = f(n)$ where f is the positive, decreasing, continuous function $f(x) = \frac{1}{x^3}$. Therefore the error R_n of the nth partial sum is less than the value of the improper integral $\int_n^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_n^{\infty} = \frac{1}{2n^2}$. Therefore we must find n such that $R_n \leq \frac{1}{2n^2} \leq .001$, that is, such that $2n^2 \geq 1000$. Letting $n = 23$ (or any higher number whose square is greater than 500) will suffice.

Problem 6.

Determine whether the following series are conditionally convergent, absolutely convergent, or divergent. Clearly show that the hypotheses of any theorems you use are satisfied.

(a) [5pts.]

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$$

Solution:
Consider the series $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi/3)}{n!}\right|$. It has positive terms and we can compare it to $\sum_{n=1}^{\infty} \frac{1}{n!}$, which we know to be convergent (either from class or by the Ratio Test). Since $\left|\frac{\cos(n\pi/3)}{n!}\right| \leq \frac{1}{n!}$ for all n, $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi/3)}{n!}\right|$ converges, and therefore $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ converges absolutely.

(b) [5pts.]

$$\sum_{n=1}^{\infty} \frac{n^2+4n+3}{5-3n^2}$$

Solution: The limit of the terms of the series is $\frac{1}{3}$, and in particular not 0. Ergo the series diverges.

(c) [5pts.]

$$\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$$

Solution:

We use the alternating series test with $b_n = \sqrt{n+1} - \sqrt{n}$. Let $f(x) = \sqrt{x+1} - \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$, so $f(x)$ is decreasing and in particular $b_{n+1} < b_n$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \\
&= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}} \\
&= 0
\end{aligned}$$

Therefore the series converges at least conditionally by the alternating series test. To test whether it converges absolutely, notice that $\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$ is a telescoping series with partial sums $s_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n} - \sqrt{n-1}) + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1$. This diverges to infinity as $n \to \infty$. Ergo the series is only conditionally convergent.

Problem 7.

(a) [10pts.] Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{n^{\frac{1}{4}}}$$

Solution: To determine R, we use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{3^{n+1}(x+4)^n}{(n+1)^{\frac{1}{4}}} \cdot \frac{n^{\frac{1}{4}}}{3^n (x+4)^n} \right| \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{\frac{1}{4}} 3|x+4| \\
&= 3|x+4|
\end{aligned}$$

The power series converges absolutely when $3|x+4| < 1$, or when $|x+4| < \frac{1}{3}$, and diverges when $|x+4| > \frac{1}{3}$. So $R = \frac{1}{3}$. We don't yet know whether the power series converges on the endpoints of its interval of convergence, $-4\frac{1}{3}$ and $-3\frac{2}{3}$. When $x = -3\frac{2}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$, which is a p-series with $p < 1$, hence divergent. When $x = -4\frac{1}{3}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{4}}}$, which can be shown to satisfy the hypotheses of the alternating series test, and consequently converges. The interval of convergence is $[-4\frac{1}{3}, -3\frac{2}{3})$.

(b) [10pts.] Recall that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Evaluate $\int \sin(x^2)dx$. (Note that this is impossible using the techniques of integration from the first half of the class. :)

Solution:

Given the known power series representation for $\sin x$, we see

$$\begin{aligned}\sin(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\end{aligned}$$

Therefore

$$\begin{aligned}\int \sin(x^2)dx &= C + \frac{1}{3}x^3 - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3) \cdot (2n+1)!}\end{aligned}$$

Problem 8.

Extra Credit. Suppose you know that $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges for $x = 1$ but diverges for $x = 7$. What, if anything, can you say about the following?

$$\sum_{n=0}^{\infty} c_n$$

$$\sum_{n=0}^{\infty} c_n 5^n$$

$$\sum_{n=0}^{\infty} c_n 2^n$$

Solution: We see the power series is centered at 3 and the radius of convergence is at least $|1-3| = 2$ and less than $|7-3| = 4$. So the series converges for $x \in [1,5)$, diverges for $x \in [7,\infty)$ and $x \in (-\infty,-1)$. We don't know how it behaves for $x \in [-1,1)$ and $x \in [5,7)$. In the first series $x = 4$ (so that $x - 3 = 1$). This series converges. For the second series $x - 3 = 5$, so $x = 8$ and the series diverges. For the third $x - 3 = 2$, so $x = 5$. We don't have enough information to determine the behavior of the series in this case.