Weekly Homework Week 2

Section 7.3

16. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\,\tan\theta\,d\theta$, $x = \sqrt{2}/3 \Rightarrow \theta = \frac{\pi}{4}$, $x = \frac{2}{3} \Rightarrow \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\sec\theta\,\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5 \sec^5\theta\,\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2\,d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3}$$

$$= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right)$$

Section 7.4

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3} \Rightarrow 4y^2-7y-12 = A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy = \left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2$$

$$= 2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2-\ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

23. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10 = A(x^2+9)+(Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10=10A \Leftrightarrow A=1$. Substituting 0 for x gives $10=9A-C \Rightarrow C=9(1)-10=-1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0=A+B \Rightarrow B=-1$. Thus,

$$\int\frac{10}{(x-1)(x^2+9)}\,dx = \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx$$

$$= \ln|x-1|-\tfrac{1}{2}\ln(x^2+9)-\tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)+C$$

In the second term we used the substitution $u=x^2+9$ and in the last term we used Formula 10.

34. $\dfrac{x^5+x-1}{x^3+1} = x^2+\dfrac{-x^2+x-1}{x^3+1} = x^2+\dfrac{-x^2+x-1}{(x+1)(x^2-x+1)} = x^2+\dfrac{-1}{x+1}$, so

$$\int\frac{x^5+x-1}{x^3+1}\,dx = \int\left(x^2-\frac{1}{x+1}\right)dx = \frac{1}{3}x^3-\ln|x+1|+C$$

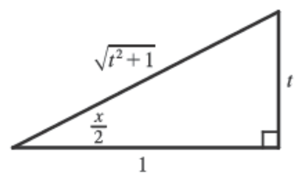
35. $\dfrac{1}{x(x^2+4)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2} \quad\Rightarrow\quad 1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$. Setting $x=0$ gives $1 = 16A$, so $A = \frac{1}{16}$. Now compare coefficients.

$$1 = \tfrac{1}{16}(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2+4) + Dx^2 + Ex$$
$$1 = \tfrac{1}{16}x^4 + \tfrac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex$$
$$1 = \left(\tfrac{1}{16} + B\right)x^4 + Cx^3 + \left(\tfrac{1}{2} + 4B + D\right)x^2 + (4C+E)x + 1$$

So $B + \frac{1}{16} = 0 \quad\Rightarrow\quad B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \quad\Rightarrow\quad D = -\frac{1}{4}$, and $4C + E = 0 \quad\Rightarrow\quad E = 0$. Thus,

$$\int \frac{dx}{x(x^2+4)^2} = \int \left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2} \right) dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln\left|x^2+4\right| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C$$
$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C$$

59. (a) If $t = \tan\left(\dfrac{x}{2}\right)$, then $\dfrac{x}{2} = \tan^{-1} t$. The figure gives



$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

(b) $\cos x = \cos\left(2\cdot\dfrac{x}{2}\right) = 2\cos^2\left(\dfrac{x}{2}\right) - 1$

$$= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

(c) $\dfrac{x}{2} = \arctan t \quad\Rightarrow\quad x = 2\arctan t \quad\Rightarrow\quad dx = \dfrac{2}{1+t^2}\,dt$

61. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 59, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t) - 4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$
$$= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right] dt \qquad \text{[using partial fractions]}$$
$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2) - 1}{\tan(x/2) + 2}\right| + C$$

Section 7.5

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \quad\Rightarrow\quad \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with $u = \arctan t$, $dv = 2t\,dt \quad\Rightarrow\quad du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1 - \frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$
$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}du$, so

$$\int \sqrt{1+e^x}\,dx = \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x} - 1\right) - \ln\left(\sqrt{1+e^x} + 1\right) + C$$

Section 6.5

11. (a) $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi (2\sin x - \sin 2x)\,dx$

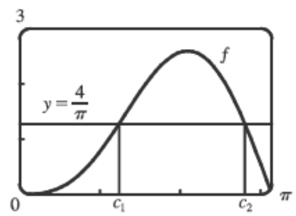
$= \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^\pi$

$= \frac{1}{\pi}\left[\left(2 + \frac{1}{2}\right) - \left(-2 + \frac{1}{2}\right)\right] = \frac{4}{\pi}$

(b) $f(c) = f_{\text{ave}} \quad \Leftrightarrow \quad 2\sin c - \sin 2c = \frac{4}{\pi} \quad \Leftrightarrow$

$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



15. Use geometric interpretations to find the values of the integrals.

$\int_0^8 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^7 f(x)\,dx + \int_7^8 f(x)\,dx$

$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + 4 + \frac{3}{2} + 2 = 9$

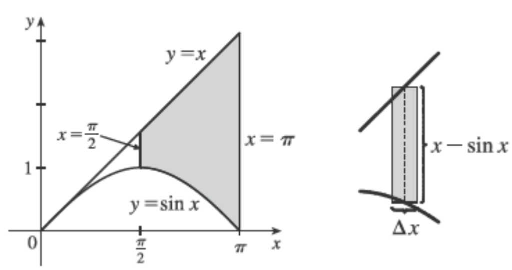
Thus, the average value of f on $[0, 8] = f_{\text{ave}} = \frac{1}{8-0}\int_0^8 f(x)\,dx = \frac{1}{8}(9) = \frac{9}{8}$.

Section 6.1

6. $A = \displaystyle\int_{\pi/2}^{\pi} (x - \sin x)\,dx = \left[\frac{x^2}{2} + \cos x\right]_{\pi/2}^{\pi}$

$= \left(\dfrac{\pi^2}{2} - 1\right) - \left(\dfrac{\pi^2}{8} + 0\right)$

$= \dfrac{3\pi^2}{8} - 1$



13. $12 - x^2 = x^2 - 6 \quad \Leftrightarrow \quad 2x^2 = 18 \quad \Leftrightarrow$

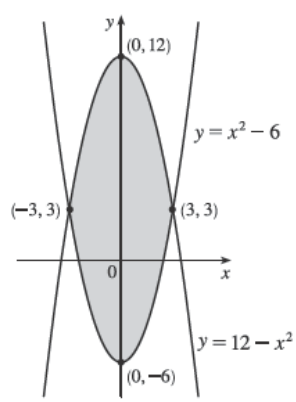
$x^2 = 9 \quad \Leftrightarrow \quad x = \pm 3$, so

$A = \displaystyle\int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6)\right] dx$

$= 2\displaystyle\int_0^3 (18 - 2x^2)\,dx \qquad$ [by symmetry]

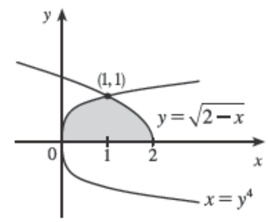
$= 2\left[18x - \frac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right]$

$= 2(36) = 72$

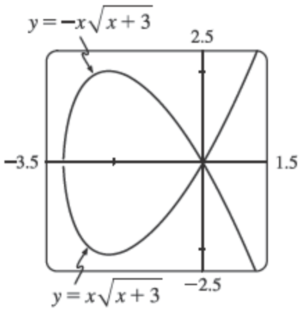


20. $y = \sqrt{2-x} \quad \Rightarrow \quad y^2 = 2 - x \quad \Leftrightarrow \quad x = 2 - y^2$, so the curves intersect when $y^4 = 2 - y^2 \quad \Leftrightarrow \quad y^4 + y^2 - 2 = 0 \quad \Leftrightarrow$ $(y^2+2)(y^2-1) = 0 \quad \Leftrightarrow \quad y = 1$ [since $y \geq 0$].

$$A = \int_0^1 [(2-y^2) - y^4)]\,dy = \left[2y - \tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_0^1$$
$$= \left(2 - \tfrac{1}{3} - \tfrac{1}{5}\right) - 0 = \tfrac{22}{15}$$



49.



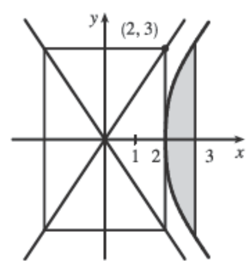
To graph this function, we must first express it as a combination of explicit functions of y; namely, $y = \pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x = -3$ to $x = 0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y = -x\sqrt{x+3}$. So the area is $A = 2\int_{-3}^{0}\left(-x\sqrt{x+3}\right)dx$. We substitute $u = x + 3$, so $du = dx$ and the limits change to 0 and 3, and we get

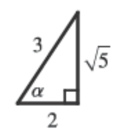
$$A = -2\int_0^3 [(u-3)\sqrt{u}\,]\,du = -2\int_0^3 (u^{3/2} - 3u^{1/2})\,du$$
$$= -2\left[\tfrac{2}{5}u^{5/2} - 2u^{3/2}\right]_0^3 = -2\left[\tfrac{2}{5}\left(3^2\sqrt{3}\right) - 2\left(3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

Section 7.3 (Additional area problem)

34. $9x^2 - 4y^2 = 36 \quad \Rightarrow \quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad \Rightarrow$

$$\text{area} = 2\int_2^3 \tfrac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$$
$$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\, \tan\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } x = 2\sec\theta, \\ dx = 2\sec\theta\,\tan\theta\,d\theta, \\ \alpha = \sec^{-1}\left(\frac{3}{2}\right)\end{array}\right]$$
$$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta\,d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right)d\theta$$
$$= 12\left[\tfrac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta|\right]_0^\alpha$$
$$= 6\left[\sec\theta\,\tan\theta - \ln|\sec\theta + \tan\theta|\right]_0^\alpha = 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)$$

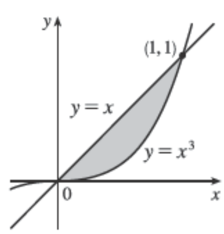


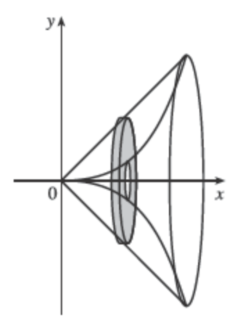


Section 6.2

7. A cross-section is a washer (annulus) with inner radius x^3 and outer radius x, so its area is $A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx$$
$$= \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{7}\right) = \tfrac{4}{21}\pi$$



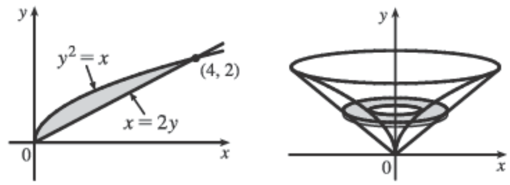


9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is

$A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4)$.

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 (4y^2 - y^4)\,dy$$

$$= \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64}{15}\pi$$



18. For $0 \le y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$, the area of which is $A_1(y) = \pi(4-1)^2 - \pi(2-1)^2$. For $2 \le y \le 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$, the area of which is $A_2(y) = \pi(4-1)^2 - \pi(y-1)^2$.

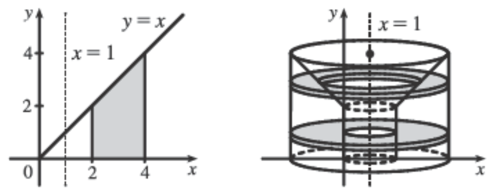
$$V = \int_0^4 A(y)\,dy = \pi \int_0^2 \left[(4-1)^2 - (2-1)^2\right] dy + \pi \int_2^4 \left[(4-1)^2 - (y-1)^2\right] dy$$

$$= \pi\left[8y\right]_0^2 + \pi \int_2^4 (8 + 2y - y^2)\,dy$$

$$= 16\pi + \pi\left[8y + y^2 - \tfrac{1}{3}y^3\right]_2^4$$

$$= 16\pi + \pi\left[\left(32 + 16 - \tfrac{64}{3}\right) - \left(16 + 4 - \tfrac{8}{3}\right)\right]$$

$$= \tfrac{76}{3}\pi$$

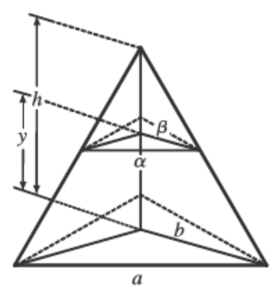


52. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y, so $a/b = \alpha/\beta \;\Rightarrow\; \alpha = a\beta/b$. Also by similar triangles, $b/h = \beta/(h - y) \Rightarrow \beta = b(h - y)/h$. These two equations imply that $\alpha = a(1 - y/h)$, and since the cross-section is an equilateral triangle, it has area

$$A(y) = \frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{3}}{2}\alpha = \frac{a^2(1 - y/h)^2}{4}\sqrt{3}, \text{ so}$$

$$V = \int_0^h A(y)\,dy = \frac{a^2\sqrt{3}}{4}\int_0^h \left(1 - \frac{y}{h}\right)^2 dy$$

$$= \frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1 - \frac{y}{h}\right)^3\right]_0^h = -\frac{\sqrt{3}}{12}a^2 h(-1) = \frac{\sqrt{3}}{12}a^2 h$$

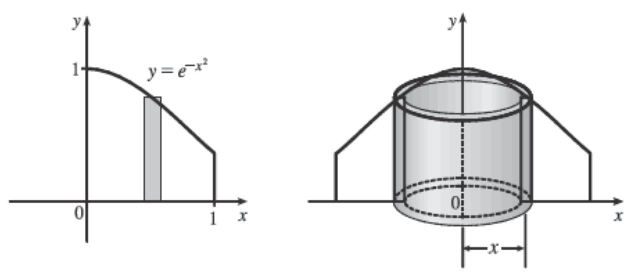


Section 6.3

5. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$.

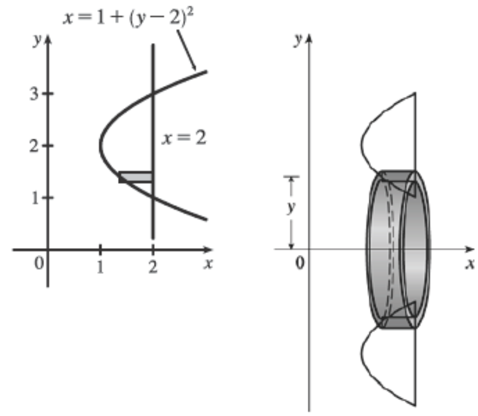
Thus, $du = 2x\,dx$, so

$V = \pi \int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.



13. The height of the shell is $2 - \left[1 + (y-2)^2\right] = 1 - (y-2)^2 = 1 - \left(y^2 - 4y + 4\right) = -y^2 + 4y - 3$.

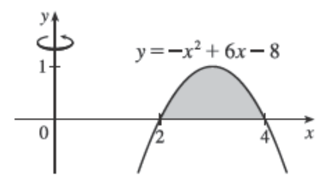
$$V = 2\pi \int_1^3 y(-y^2 + 4y - 3)\,dy$$
$$= 2\pi \int_1^3 (-y^3 + 4y^2 - 3y)\,dy$$
$$= 2\pi \left[-\tfrac{1}{4}y^4 + \tfrac{4}{3}y^3 - \tfrac{3}{2}y^2\right]_1^3$$
$$= 2\pi \left[\left(-\tfrac{81}{4} + 36 - \tfrac{27}{2}\right) - \left(-\tfrac{1}{4} + \tfrac{4}{3} - \tfrac{3}{2}\right)\right]$$
$$= 2\pi\left(\tfrac{8}{3}\right) = \tfrac{16}{3}\pi$$



31. $\int_0^1 2\pi(3-y)(1-y^2)\,dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

37. Use shells:

$$V = \int_2^4 2\pi x(-x^2 + 6x - 8)\,dx = 2\pi \int_2^4 (-x^3 + 6x^2 - 8x)\,dx$$
$$= 2\pi\left[-\tfrac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4$$
$$= 2\pi[(-64 + 128 - 64) - (-4 + 16 - 16)]$$
$$= 2\pi(4) = 8\pi$$



Section 7.4 (Additional volume problem)

66. (a) We use disks, so the volume is $V = \pi \displaystyle\int_0^1 \left[\frac{1}{x^2 + 3x + 2}\right]^2 dx = \pi \int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} \Rightarrow$

$1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x = -1$, giving $B = 1$, then set $x = -2$, giving $D = 1$. Now equating coefficients of x^3 gives $A = -C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \;\Rightarrow\; A = -2 \;\Rightarrow\; C = 2$. So the expression becomes

$$V = \pi \int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}\right] dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right| - \frac{1}{x+1} - \frac{1}{x+2}\right]_0^1$$
$$= \pi\left[\left(2\ln\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(2\ln 2 - 1 - \tfrac{1}{2}\right)\right] = \pi\left(2\ln\tfrac{3/2}{2} + \tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3} + \ln\tfrac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi \displaystyle\int_0^1 \frac{x\,dx}{x^2 + 3x + 2} = 2\pi \int_0^1 \frac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \;\Rightarrow\; x = (A+B)x + 2A + B$. So $A + B = 1$ and $2A + B = 0 \;\Rightarrow\; A = -1$ and $B = 2$. So the volume is

$$2\pi \int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right] dx = 2\pi\Big[-\ln|x+1| + 2\ln|x+2|\Big]_0^1$$
$$= 2\pi(-\ln 2 + 2\ln 3 + \ln 1 - 2\ln 2) = 2\pi(2\ln 3 - 3\ln 2) = 2\pi \ln\tfrac{9}{8}$$

Section 6.4

9. (a) If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m.

Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

(b) $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

In Exercises 13–20, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

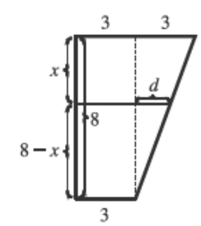
(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2}\cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4}\cdot 625 = \frac{1875}{4}$ ft-lb.

23. Let x measure depth (in feet) below the spout at the top of the tank. A horizontal disk-shaped "slice" of water Δx ft thick and lying at coordinate x has radius $\frac{3}{8}(16 - x)$ ft $(\star)$ and volume $\pi r^2\Delta x = \pi\cdot\frac{9}{64}(16 - x)^2\,\Delta x$ ft^3. It weighs about $(62.5)\frac{9\pi}{64}(16 - x)^2\,\Delta x$ lb and must be lifted x ft by the pump, so the work needed to pump it out is about $(62.5)x\,\frac{9\pi}{64}(16 - x)^2\,\Delta x$ ft-lb. The total work required is

$$W \approx \int_0^8 (62.5)x\,\tfrac{9\pi}{64}(16 - x)^2\,dx = (62.5)\tfrac{9\pi}{64}\int_0^8 x(256 - 32x + x^2)\,dx$$

$$= (62.5)\tfrac{9\pi}{64}\int_0^8 (256x - 32x^2 + x^3)\,dx = (62.5)\tfrac{9\pi}{64}\left[128x^2 - \tfrac{32}{3}x^3 + \tfrac{1}{4}x^4\right]_0^8$$

$$= (62.5)\frac{9\pi}{64}\left(\frac{11,264}{3}\right) = 33,000\pi \approx 1.04\times 10^5 \text{ ft-lb}$$



$(\star)$ From similar triangles, $\dfrac{d}{8 - x} = \dfrac{3}{8}$.

So $r = 3 + d = 3 + \frac{3}{8}(8 - x)$

$= \dfrac{3(8)}{8} + \dfrac{3}{8}(8 - x)$

$= \frac{3}{8}(16 - x)$