Weekly Homework Week 3

Section 8.1

8. $y^2 = 4(x+4)^3,\ y > 0 \quad\Rightarrow\quad y = 2(x+4)^{3/2} \quad\Rightarrow\quad dy/dx = 3(x+4)^{1/2} \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37$. So

$$L = \int_0^2 \sqrt{9x+37}\,dx \quad \begin{bmatrix} u = 9x+37, \\ du = 9\,dx \end{bmatrix} \quad = \int_{37}^{55} u^{1/2}\left(\tfrac{1}{9}\,du\right) = \tfrac{1}{9}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \tfrac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right).$$

10. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2} \quad\Rightarrow\quad \dfrac{dx}{dy} = \tfrac{1}{2}y^3 - \tfrac{1}{2}y^{-3} \quad\Rightarrow$

$1 + (dx/dy)^2 = 1 + \tfrac{1}{4}y^6 - \tfrac{1}{2} + \tfrac{1}{4}y^{-6} = \tfrac{1}{4}y^6 + \tfrac{1}{2} + \tfrac{1}{4}y^{-6} = \left(\tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}\right)^2$. So

$$L = \int_1^2 \sqrt{\left(\tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}\right)^2}\,dy = \int_1^2 \left(\tfrac{1}{2}y^3 + \tfrac{1}{2}y^{-3}\right)dy = \left[\tfrac{1}{8}y^4 - \tfrac{1}{4}y^{-2}\right]_1^2 = \left(2 - \tfrac{1}{16}\right) - \left(\tfrac{1}{8} - \tfrac{1}{4}\right)$$

$$= 2 + \tfrac{1}{16} = \tfrac{33}{16}.$$

16. $y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right) \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1-2x}{2\sqrt{x-x^2}} + \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \sqrt{\dfrac{1-x}{x}} \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1-x}{x} = \dfrac{1}{x}$. The curve has endpoints $(0,0)$ and $\left(1, \frac{\pi}{2}\right)$, so $L = \displaystyle\int_0^1 \sqrt{\frac{1}{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = 2.$

20. $x^2 = (y-4)^3 \quad\Rightarrow\quad x = (y-4)^{3/2}$ [for $x > 0$] $\quad\Rightarrow\quad dx/dy = \tfrac{3}{2}(y-4)^{1/2} \quad\Rightarrow$

$1 + (dx/dy)^2 = 1 + \tfrac{9}{4}(y-4) = \tfrac{9}{4}y - 8$. So

$$L = \int_5^8 \sqrt{\tfrac{9}{4}y - 8}\,dy = \int_{13/4}^{10} \sqrt{u}\ \left(\tfrac{4}{9}\,du\right) \quad \begin{bmatrix} u = \frac{9}{4}y - 8, \\ du = \frac{9}{4}\,dy \end{bmatrix} \quad = \tfrac{4}{9}\left[\tfrac{2}{3}u^{3/2}\right]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right] \quad \left[\text{or } \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)\right]$$

Section 8.2

5. $y = x^3 \quad\Rightarrow\quad y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx]$$

$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

11. $x = \tfrac{1}{3}(y^2+2)^{3/2} \quad\Rightarrow\quad dx/dy = \tfrac{1}{2}(y^2+2)^{1/2}(2y) = y\sqrt{y^2+2} \quad\Rightarrow\quad 1 + (dx/dy)^2 = 1 + y^2(y^2+2) = (y^2+1)^2.$

So $S = 2\pi\int_1^2 y(y^2+1)\,dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}.$

15. $x = \sqrt{a^2 - y^2} \quad\Rightarrow\quad dx/dy = \tfrac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \quad\Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \quad\Rightarrow$$

$$S = \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\,\frac{a}{\sqrt{a^2 - y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\left[\,y\,\right]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2.$$

Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

16. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x} \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{x^2}{4} - \dfrac{1}{2} + \dfrac{1}{4x^2} = \dfrac{x^2}{4} + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2$. So

$$S = \int_1^2 2\pi x \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\,dx = 2\pi \int_1^2 x\left(\frac{x}{2} + \frac{1}{2x}\right)dx = \pi \int_1^2 (x^2 + 1)\,dx = \pi\left[\tfrac{1}{3}x^3 + x\right]_1^2$$
$$= \pi\left[\left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)\right] = \tfrac{10}{3}\pi$$

33. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

$$S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\pi \int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\,dx$$
$$= 4\pi \int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right)dx$$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \dfrac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right)dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}}\right)dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2\left(\frac{\pi}{2}\right) = 4\pi^2 r^2$.

Section 7.8

9. $\int_2^\infty e^{-5p}\,dp = \lim\limits_{t\to\infty} \int_2^t e^{-5p}\,dp = \lim\limits_{t\to\infty}\left[-\frac{1}{5}e^{-5p}\right]_2^t = \lim\limits_{t\to\infty}\left(-\frac{1}{5}e^{-5t} + \frac{1}{5}e^{-10}\right) = 0 + \frac{1}{5}e^{-10} = \frac{1}{5}e^{-10}$. Convergent

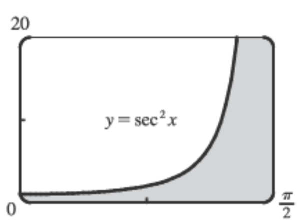
16. $I = \int_{-\infty}^{\infty} \cos \pi t\,dt = I_1 + I_2 = \int_{-\infty}^{0} \cos \pi t\,dt + \int_0^\infty \cos \pi t\,dt$, but $I_1 = \lim\limits_{s\to-\infty}\left[\dfrac{1}{\pi}\sin \pi t\right]_s^0 = \lim\limits_{s\to-\infty}\left(-\dfrac{1}{\pi}\sin \pi t\right)$ and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

22. $I = \int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx = I_1 + I_2 = \int_{-\infty}^{0} x^3 e^{-x^4}\,dx + \int_0^\infty x^3 e^{-x^4}\,dx$. Now

$$I_2 = \lim_{t\to\infty}\int_0^t x^3 e^{-x^4}\,dx = \lim_{t\to\infty}\int_0^{t^4} e^{-u}\left(\tfrac{1}{4}\,du\right) \qquad \left[\begin{array}{l} u = x^4, \\ du = 4x^3\,dx \end{array}\right]$$
$$= \tfrac{1}{4}\lim_{t\to\infty}\left[-e^{-u}\right]_0^{t^4} = \tfrac{1}{4}\lim_{t\to\infty}\left(-e^{-t^4} + 1\right) = \tfrac{1}{4}(0 + 1) = \tfrac{1}{4}.$$

Since $f(x) = x^3 e^{-x^4}$ is an odd function, $I_1 = -\frac{1}{4}$, and hence, $I = 0$. Convergent

45.



Area $= \int_0^{\pi/2} \sec^2 x\,dx = \lim\limits_{t\to(\pi/2)^-} \int_0^t \sec^2 x\,dx = \lim\limits_{t\to(\pi/2)^-}\left[\tan x\right]_0^t$

$= \lim\limits_{t\to(\pi/2)^-}(\tan t - 0) = \infty$

Infinite area

49. For $x > 0$, $\dfrac{x}{x^3 + 1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3 + 1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3 + 1}\,dx = \int_0^1 \frac{x}{x^3 + 1}\,dx + \int_1^\infty \frac{x}{x^3 + 1}\,dx$ is also convergent.

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} \quad \begin{bmatrix} u=\sqrt{x},\, x=u^2, \\ dx = 2u\,du \end{bmatrix} \quad = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C, \text{ so}$$

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$