**8.** $f(x) = \frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when

$\left|-2x^2\right| < 1 \quad \Rightarrow \quad \left|x^2\right| < \frac{1}{2} \quad \Rightarrow \quad |x| < \frac{1}{\sqrt{2}}$, so $R = \frac{1}{\sqrt{2}}$ and $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

**11.** $f(x) = \frac{3}{x^2-x-2} = \frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad \Rightarrow \quad 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and $x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1-(x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented $f$ as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in (-1, 1) = I$.

**13.** (a) $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right) = -\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} \quad \text{[from Theorem 2(i)]} \quad = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable $n$ by 1, and then *increased* each occurrence of $n$ in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \text{ with } R = 1.$$

(c) $f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$$

To write the power series with $x^n$ rather than $x^{n+2}$, we will *decrease* each occurrence of $n$ in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R = 1$.

**15.** $f(x) = \ln(5-x) = -\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$.

**25.** $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t \sum\limits_{n=0}^{\infty} (t^8)^n = \sum\limits_{n=0}^{\infty} t^{8n+1} \quad \Rightarrow \quad \displaystyle\int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1 \quad \Leftrightarrow \quad |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1-t^8}\,dt$ also has $R = 1$.

**15.**

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(2)$ |
|---|---|---|
| 0 | $\ln x$ | $\ln 2$ |
| 1 | $1/x$ | $1/2$ |
| 2 | $-1/x^2$ | $-1/2^2$ |
| 3 | $2/x^3$ | $2/2^3$ |
| 4 | $-6/x^4$ | $-6/2^4$ |
| 5 | $24/x^5$ | $24/2^5$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

$$\begin{aligned} f(x) = \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{\ln 2}{0!}(x-2)^0 + \frac{1}{1!\,2^1}(x-2)^1 + \frac{-1}{2!\,2^2}(x-2)^2 + \frac{2}{3!\,2^3}(x-2)^3 \\ &\qquad + \frac{-6}{4!\,2^4}(x-2)^4 + \frac{24}{5!\,2^5}(x-2)^5 + \cdots \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{n!\,2^n}(x-2)^n \\ &= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\,2^n}(x-2)^n \end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(-1)^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-2)n}{(n+1)2}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{|x-2|}{2}$$

$$= \frac{|x-2|}{2} < 1 \quad \text{for convergence, so } |x-2| < 2 \text{ and } R = 2.$$

**19.**

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(\pi)$ |
|---|---|---|
| 0 | $\cos x$ | $-1$ |
| 1 | $-\sin x$ | 0 |
| 2 | $-\cos x$ | 1 |
| 3 | $\sin x$ | 0 |
| 4 | $\cos x$ | $-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

$$\begin{aligned} f(x) = \cos x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k \\ &= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\pi|^{2n}}\right] \\ &= \lim_{n\to\infty} \frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty. \end{aligned}$$

**31.** $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \Rightarrow \quad e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{(2x)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^n}{n!}$, so $f(x) = e^x + e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!}x^n + \sum\limits_{n=0}^{\infty} \dfrac{2^n}{n!}x^n = \sum\limits_{n=0}^{\infty} \dfrac{2^n+1}{n!}x^n$,

$R = \infty$.

**44.** $1/\sqrt[10]{e} = e^{-1/10}$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so

$e^{-1/10} = 1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} - \frac{(1/10)^5}{5!} + \cdots$. Now

$1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} \approx 0.90484$ and subtracting $\frac{(1/10)^5}{5!} \approx 8.3 \times 10^{-8}$ does not affect the fifth decimal place, so $e^{-1/10} \approx 0.90484$ by the Alternating Series Estimation Theorem.

**63.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$, by (11).

**14.**

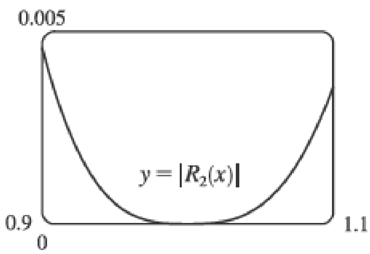
| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ |
|---|---|---|
| 0 | $x^{-2}$ | 1 |
| 1 | $-2x^{-3}$ | $-2$ |
| 2 | $6x^{-4}$ | 6 |
| 3 | $-24x^{-5}$ | |

(a) $f(x) = x^{-2} \approx T_2(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 = 1 - 2(x-1) + 3(x-1)^2$

(b) $|R_2(x)| \le \frac{M}{3!}|x-1|^3$, where $|f'''(x)| \le M$. Now $0.9 \le x \le 1.1 \Rightarrow |x-1| \le 0.1 \Rightarrow |x-1|^3 \le 0.001$. Since $f'''(x)$ is decreasing on $[0.9, 1.1]$, we can take $M = |f'''(0.9)| = \frac{24}{(0.9)^5}$, so

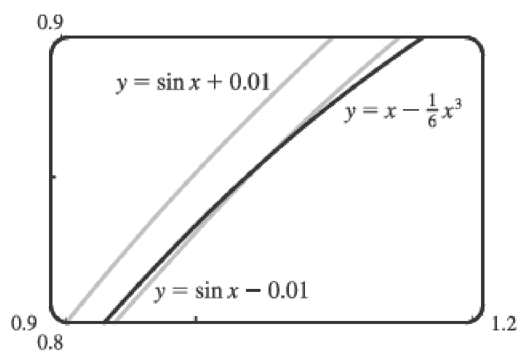
$|R_2(x)| \le \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049} \approx 0.006\,774\,04$.

(c)




From the graph of $|R_2(x)| = \left|x^{-2} - T_2(x)\right|$, it seems that the error is less than $0.0046$ on $[0.9, 1.1]$.

**27.** $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow$ $\left|x^5\right| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for $x$ is $-1.037 < x < 1.037$.



**31.** Let $s(t)$ be the position function of the car, and for convenience set $s(0) = \mathbf{0}$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance traveled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of $2\ \mathrm{m/s^2}$ for that long (if it did, its final speed would be $140\ \mathrm{m/s} \approx 313\ \mathrm{mi/h}$!).