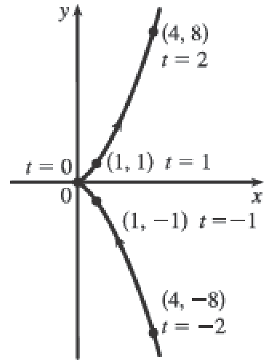
10. $x = t^2$, $y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

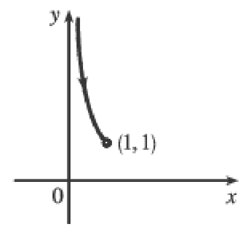
(b) $y = t^3 \quad \Rightarrow \quad t = \sqrt[3]{y} \quad \Rightarrow \quad x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$. $\quad t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0$.



13. (a) $x = \sin t$, $y = \csc t$, $0 < t < \frac{\pi}{2}$. $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$.

For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

(b)



28. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $\quad x = t^2 - 2t = t(t - 2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = \sin 2t$ has period $2\pi/2 = \pi$. Note that
$y(t + 2\pi) = \sin[t + 2\pi + \sin 2(t + 2\pi)] = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)$, so y has period 2π.
These equations match graph II since x cycles through the values -1 to 1 twice as y cycles through those values once.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

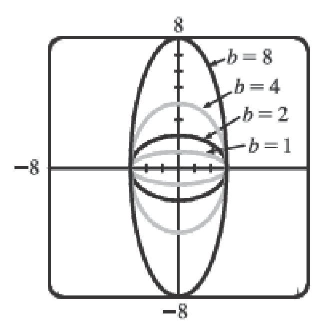
(e) $x = t + \sin 4t$, $\ y = t^2 + \cos 3t$. $\quad$ As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = \dfrac{\sin 2t}{4 + t^2}$, $\ y = \dfrac{\cos 2t}{4 + t^2}$. $\quad$ As $t \to \infty$, x and y both approach 0. These equations are matched with graph III.

34. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a \sin t$ and $y = b \cos t$ with $0 \leq t \leq 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

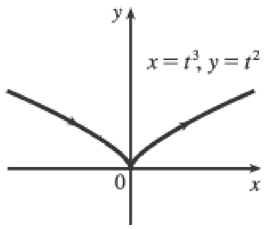
(b) The equations are $x = 3 \sin t$ and $y = b \cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.



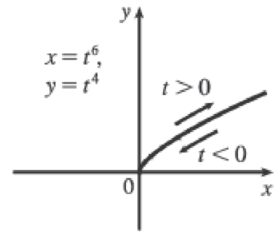
37. (a) $x = t^3 \Rightarrow t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \Rightarrow t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

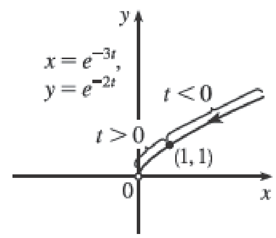
Since $x = t^6 \geq 0$, we only get the right half of the curve $y = x^{2/3}$.



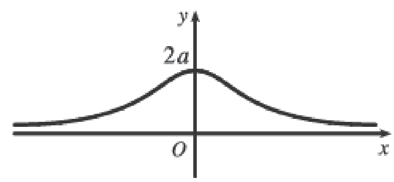
(c) $x = e^{-3t} = (e^{-t})^3$ [so $e^{-t} = x^{1/3}$],

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.

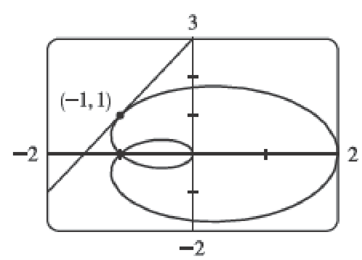


43. $C = (2a \cot \theta, 2a)$, so the x-coordinate of P is $x = 2a \cot \theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a \sin \theta$ and $A = ((2a \sin \theta) \cos \theta, (2a \sin \theta) \sin \theta)$. Thus, the y-coordinate of P is $y = 2a \sin^2 \theta$.



10. $x = \cos t + \cos 2t$, $y = \sin t + \sin 2t$; $(-1, 1)$.

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t}$. To find the value of t corresponding to the point $(-1, 1)$, solve $x = -1 \Rightarrow \cos t + \cos 2t = -1 \Rightarrow$ $\cos t + 2\cos^2 t - 1 = -1 \Rightarrow \cos t\,(1 + 2\cos t) = 0 \Rightarrow \cos t = 0$ or $\cos t = -\frac{1}{2}$. The interval $[0, 2\pi]$ gives the complete curve, so we need only find the values of t in this interval. Thus, $t = \frac{\pi}{2}$ or $t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$. Checking $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ in the equation for y, we find that $t = \frac{\pi}{2}$ corresponds to $(-1, 1)$. The slope of the tangent at $(-1, 1)$ with $t = \frac{\pi}{2}$ is $\dfrac{0 - 2}{-1 - 0} = 2$. An equation of the tangent is therefore $y - 1 = 2(x + 1)$, or $y = 2x + 3$.



13. $x = e^t$, $y = te^{-t} \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-te^{-t} + e^{-t}}{e^t} = \dfrac{e^{-t}(1 - t)}{e^t} = e^{-2t}(1 - t) \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{e^{-2t}(-1) + (1 - t)(-2e^{-2t})}{e^t} = \frac{e^{-2t}(-1 - 2 + 2t)}{e^t} = e^{-3t}(2t - 3).$$

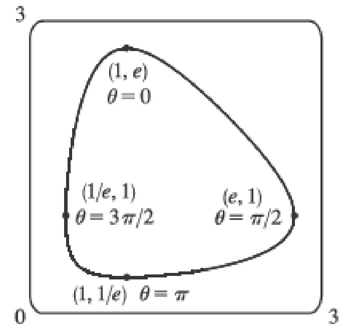
The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

20. $x = e^{\sin\theta}$, $y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \quad\Leftrightarrow\quad \sin\theta = 0 \quad\Leftrightarrow\quad \theta = 0 \text{ or } \pi \quad\Leftrightarrow$

$(x, y) = (1, e)$ or $(1, 1/e)$. $\quad \dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \quad\Leftrightarrow\quad \cos\theta = 0 \quad\Leftrightarrow$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad (x, y) = (e, 1)$ or $(1/e, 1)$. The curve has horizontal tangents at $(1, e)$ and $(1, 1/e)$, and vertical tangents at $(e, 1)$ and $(1/e, 1)$.

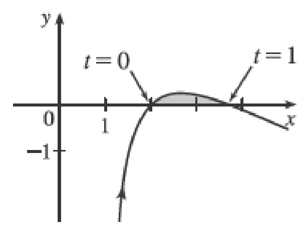


31. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

33. The curve $x = 1 + e^t$, $y = t - t^2 = t(1-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$. The shaded area is given by



$$\int_{x=2}^{x=1+e}(y_T - y_B)\,dx = \int_{t=0}^{t=1}[y(t) - 0]\,x'(t)\,dt = \int_0^1 (t - t^2)e^t\,dt$$

$$= \int_0^1 te^t\,dt - \int_0^1 t^2e^t\,dt = \int_0^1 te^t\,dt - \left[t^2e^t\right]_0^1 + 2\int_0^1 te^t\,dt \qquad \text{[Formula 97 or parts]}$$

$$= 3\int_0^1 te^t\,dt - (e - 0) = 3\left[(t-1)e^t\right]_0^1 - e \qquad \text{[Formula 96 or parts]}$$

$$= 3[0 - (-1)] - e = 3 - e$$

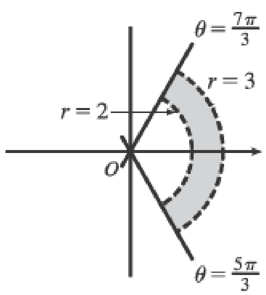
34. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\,\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$. Now

$$\int\sin^4\theta\,\cos^2\theta\,d\theta = \int\sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$

$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\,\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

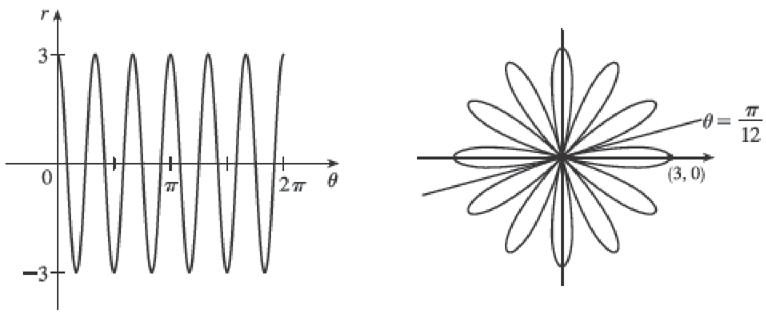
11. $2 < r < 3,\ \frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



17. $r = 2\cos\theta \quad \Rightarrow \quad r^2 = 2r\cos\theta \quad \Leftrightarrow \quad x^2 + y^2 = 2x \quad \Leftrightarrow \quad x^2 - 2x + 1 + y^2 = 1 \quad \Leftrightarrow \quad (x-1)^2 + y^2 = 1$, a circle of radius 1 centered at $(1, 0)$. The first two equations are actually equivalent since $r^2 = 2r\cos\theta \quad \Rightarrow \quad r(r - 2\cos\theta) = 0 \quad \Rightarrow$ $r = 0$ or $r = 2\cos\theta$. But $r = 2\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 2\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 2\cos\theta$).

25. $x^2 + y^2 = 2cx \quad \Leftrightarrow \quad r^2 = 2cr\cos\theta \quad \Leftrightarrow \quad r^2 - 2cr\cos\theta = 0 \quad \Leftrightarrow \quad r(r - 2c\cos\theta) = 0 \quad \Leftrightarrow \quad r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

38. $r = 3\cos 6\theta$



54. (a) $r = \sqrt{\theta}$, $0 \le \theta \le 16\pi$. r increases as θ increases and there are eight full revolutions. The graph must be either II or V. When $\theta = 2\pi$, $r = \sqrt{2\pi} \approx 2.5$ and when $\theta = 16\pi$, $r = \sqrt{16\pi} \approx 7$, so the last revolution intersects the polar axis at approximately 3 times the distance that the first revolution intersects the polar axis, which is depicted in graph V.

(b) $r = \theta^2$, $0 \le \theta \le 16\pi$. See part (a). This is graph II.

(c) $r = \cos(\theta/3)$. $0 \le \frac{\theta}{3} \le 2\pi \quad \Rightarrow \quad 0 \le \theta \le 6\pi$, so this curve will repeat itself every 6π radians.
$\cos\left(\frac{\theta}{3}\right) = 0 \quad \Rightarrow \quad \frac{\theta}{3} = \frac{\pi}{2} + \pi n \quad \Rightarrow \quad \theta = \frac{3\pi}{2} + 3\pi n$, so there will be two "pole" values, $\frac{3\pi}{2}$ and $\frac{9\pi}{2}$.
This is graph VI.

(d) $r = 1 + 2\cos\theta$ is a limaçon [see Exercise 53(a)] with $c = 2$. This is graph III.

(e) Since $-1 \le \sin 3\theta \le 1$, $1 \le 2 + \sin 3\theta \le 3$, so $r = 2 + \sin 3\theta$ is never 0; that is, the curve never intersects the pole. This is graph I.

(f) $r = 1 + 2\sin 3\theta$. Solving $r = 0$ will give us many "pole" values, so this is graph IV.

57. $r = 1/\theta \quad \Rightarrow \quad x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \dfrac{-\pi}{1} = -\pi$.

3. $r^2 = 9\sin 2\theta,\ r \geq 0,\ 0 \leq \theta \leq \pi/2.$

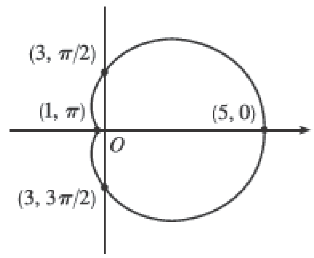
$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \tfrac{1}{2}(9\sin 2\theta)\, d\theta = \tfrac{9}{2}\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\pi/2} = -\tfrac{9}{4}(-1-1) = \tfrac{9}{2}$$

11. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(3+2\cos\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{2\pi}(9+12\cos\theta+4\cos^2\theta)\, d\theta$

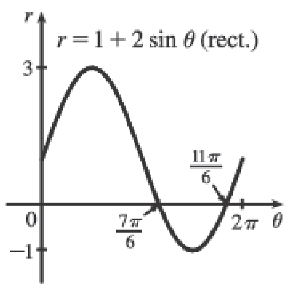
$= \tfrac{1}{2}\int_0^{2\pi}\left[9+12\cos\theta+4\cdot\tfrac{1}{2}(1+\cos 2\theta)\right] d\theta$

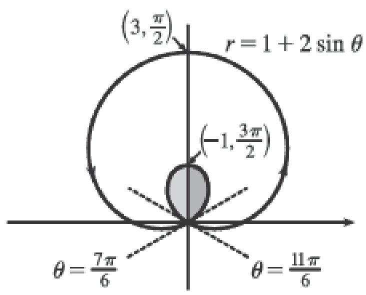
$= \tfrac{1}{2}\int_0^{2\pi}(11+12\cos\theta+2\cos 2\theta)\, d\theta = \tfrac{1}{2}\left[11\theta+12\sin\theta+\sin 2\theta\right]_0^{2\pi}$

$= \tfrac{1}{2}(22\pi) = 11\pi$



21.





This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1+2\sin\theta)^2\, d\theta = \int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right) d\theta = \int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

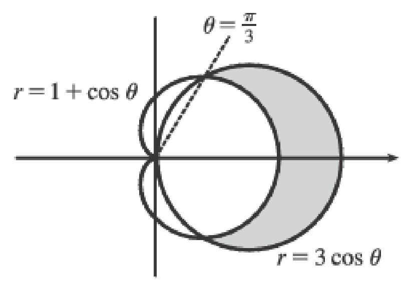
27. $3\cos\theta = 1+\cos\theta \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.

$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$

$= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\, d\theta = \int_0^{\pi/3}[4(1+\cos 2\theta) - 2\cos\theta - 1]\, d\theta$

$= \int_0^{\pi/3}(3+4\cos 2\theta - 2\cos\theta)\, d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3}$

$= \pi + \sqrt{3} - \sqrt{3} = \pi$

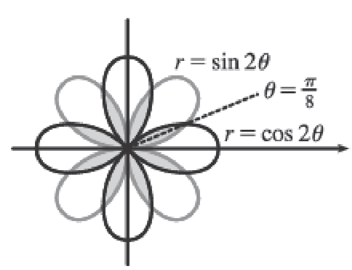


31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$

$\theta = \frac{\pi}{8} \quad\Rightarrow$

$A = 8\cdot 2\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\, d\theta = 8\int_0^{\pi/8}\tfrac{1}{2}(1-\cos 4\theta)\, d\theta$

$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\right) = \tfrac{\pi}{2} - 1$



47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta$

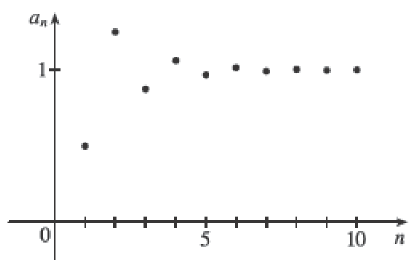
Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}\left[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}\right] = \tfrac{8}{3}\left[(\pi^2+1)^{3/2} - 1\right]$$

17. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$. The numerator of the nth term is n^2 and its denominator is $n+1$. Including the alternating signs, we get $a_n = (-1)^{n+1}\dfrac{n^2}{n+1}$.

21.

n	$a_n = 1 + \left(-\frac{1}{2}\right)^n$
1	0.5000
2	1.2500
3	0.8750
4	1.0625
5	0.9688
6	1.0156
7	0.9922
8	1.0039
9	0.9980
10	1.0010



It appears that $\lim_{n\to\infty} a_n = 1$.

$\lim_{n\to\infty} \left(1 + \left(-\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 1 + 0 = 1$ since $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ by (9).

23. $a_n = 1 - (0.2)^n$, so $\lim_{n\to\infty} a_n = 1 - 0 = 1$ by (9). Converges

35. $a_n = \cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n \to \infty$. The terms take on values between -1 and 1.

39. $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}\cdot\dfrac{e^{-n}}{e^{-n}} = \dfrac{1 + e^{-2n}}{e^n - e^{-n}} \to 0$ as $n \to \infty$ because $1 + e^{-2n} \to 1$ and $e^n - e^{-n} \to \infty$. Converges

42. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$ because ln is continuous. Converges

73. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.