15. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (11.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

18. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$ is a geometric series with ratio $\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series converges to $\dfrac{a}{1-r} = \dfrac{4}{1-3/4} = 16$.

32. $\sum_{n=1}^{\infty} \dfrac{1+3^n}{2^n} = \sum_{n=1}^{\infty} \left(\dfrac{1}{2^n} + \dfrac{3^n}{2^n}\right) = \sum_{n=1}^{\infty} \left[\left(\dfrac{1}{2}\right)^n + \left(\dfrac{3}{2}\right)^n\right] = \sum_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\dfrac{3}{2}\right)^n$. The first series is a convergent geometric series ($|r| = \frac{1}{2} < 1$), but the second series is a divergent geometric series ($|r| = \frac{3}{2} \geq 1$), so the original series is divergent.

39. $\sum_{n=1}^{\infty} \arctan n$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$.

41. $\sum_{n=1}^{\infty} \dfrac{1}{e^n} = \sum_{n=1}^{\infty} \left(\dfrac{1}{e}\right)^n$ is a geometric series with first term $a = \dfrac{1}{e}$ and ratio $r = \dfrac{1}{e}$. Since $|r| = \dfrac{1}{e} < 1$, the series converges to $\dfrac{1/e}{1-1/e} = \dfrac{1/e}{1-1/e} \cdot \dfrac{e}{e} = \dfrac{1}{e-1}$. By Example 7, $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$. Thus, by Theorem 8(ii),

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \frac{1}{e-1} + \frac{e-1}{e-1} = \frac{e}{e-1}.$$

46. For the series $\sum_{n=1}^{\infty} \left(\cos \dfrac{1}{n^2} - \cos \dfrac{1}{(n+1)^2}\right)$,

$$s_n = \sum_{i=1}^{n} \left(\cos \frac{1}{i^2} - \cos \frac{1}{(i+1)^2}\right) = \left(\cos 1 - \cos \frac{1}{4}\right) + \left(\cos \frac{1}{4} - \cos \frac{1}{9}\right) + \cdots + \left(\cos \frac{1}{n^2} - \cos \frac{1}{(n+1)^2}\right)$$

$$= \cos 1 - \cos \frac{1}{(n+1)^2} \quad \text{[telescoping series]}$$

Thus, $\sum_{n=1}^{\infty} \left(\cos \dfrac{1}{n^2} - \cos \dfrac{1}{(n+1)^2}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\cos 1 - \cos \dfrac{1}{(n+1)^2}\right) = \cos 1 - \cos 0 = \cos 1 - 1$.

Converges

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots = \sum_{n=1}^{\infty} \dfrac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

19. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3} = \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ since $\frac{\ln 1}{1} = 0$. The function $f(x) = \frac{\ln x}{x^3}$ is continuous and positive on $[2, \infty)$.

$f'(x) = \frac{x^3(1/x) - (\ln x)(3x^2)}{(x^3)^2} = \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3\ln x}{x^4} < 0 \quad\Leftrightarrow\quad 1 - 3\ln x < 0 \quad\Leftrightarrow\quad \ln x > \frac{1}{3} \quad\Leftrightarrow$

$x > e^{1/3} \approx 1.4$, so f is decreasing on $[2, \infty)$, and the Integral Test applies.

$\int_2^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x^3}\,dx \overset{(\star)}{=} \lim_{t\to\infty} \left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{4t^2}(2\ln t + 1) + \frac{1}{4}\right] \overset{(\star\star)}{=} \frac{1}{4}$, so the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges.

$(\star)$: $u = \ln x,\ dv = x^{-3}\,dx \quad\Rightarrow\quad du = (1/x)\,dx,\ v = -\frac{1}{2}x^{-2}$, so

$\int \frac{\ln x}{x^3}\,dx = -\frac{1}{2}x^{-2}\ln x - \int -\frac{1}{2}x^{-2}(1/x)\,dx = -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,dx = -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C.$

$(\star\star)$: $\lim_{t\to\infty}\left(-\frac{2\ln t + 1}{4t^2}\right) \overset{H}{=} -\lim_{t\to\infty}\frac{2/t}{8t} = -\frac{1}{4}\lim_{t\to\infty}\frac{1}{t^2} = 0.$

34. (a) $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{1^2}$ [subtract a_1] $= \frac{\pi^2}{6} - 1$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) = \frac{\pi^2}{6} - \frac{49}{36}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$

36. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037.$

$R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3\,(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx \quad\Rightarrow\quad s_{10} + \frac{1}{3(11)^3} \le s \le s_{10} + \frac{1}{3(10)^3} \quad\Rightarrow$

$1.082037 + 0.000250 = 1.082287 \le s \le 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) The estimate in part (b) is $s \approx 1.08233$ with error ≤ 0.00005. The exact value given in Exercise 35 is $\pi^4/90 \approx 1.082323$. The difference is less than 0.00001.

(d) $R_n \le \int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \quad\Rightarrow\quad \frac{1}{3n^3} < \frac{1}{10^5} \quad\Rightarrow\quad 3n^3 > 10^5 \quad\Rightarrow\quad n > \sqrt[3]{(10)^5/3} \approx 32.2$,

that is, for $n > 32$.

4. $\dfrac{n^3}{n^4-1} > \dfrac{n^3}{n^4} = \dfrac{1}{n}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{n^3}{n^4-1}$ diverges by comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$ (the harmonic series).

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^2 = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} e^{-n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{e^n}$ is a convergent geometric series $\left[|r| = \frac{1}{e} < 1\right]$, the series $\sum\limits_{n=1}^{\infty} \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ also converges.

30. $\dfrac{n!}{n^n} = \dfrac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n}{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n} \leq \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot 1 \cdot 1 \cdot \cdots \cdot 1$ for $n \geq 2$, so since $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2}$ converges $[p = 2 > 1]$, $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges also by the Comparison Test.

7. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{3n-1}{2n+1} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{3-1/n}{2+1/n} = \dfrac{3}{2} \neq 0$. Since $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

13. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} e^{2/n} = e^0 = 1$, so $\lim\limits_{n\to\infty} (-1)^{n-1} e^{2/n}$ does not exist. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$ diverges by the Test for Divergence.

23. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^6} < \dfrac{1}{n^6}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^6} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \dfrac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

4. $b_n = \dfrac{n}{n^2+4} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+4}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \dfrac{1}{n}$ to get

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{1/n}{n/(n^2+4)} = \lim\limits_{n\to\infty}\dfrac{n^2+4}{n^2} = \lim\limits_{n\to\infty}\dfrac{1+4/n^2}{1} = 1 > 0$, so $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+4}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+4}$ is conditionally convergent.

8. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)!}{100^{n+1}}\cdot\dfrac{100^n}{n!}\right] = \lim\limits_{n\to\infty}\dfrac{n+1}{100} = \infty$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n!}{100^n}$ diverges by the Ratio Test.

12. $\left|\dfrac{\sin 4n}{4^n}\right| \leq \dfrac{1}{4^n}$, so $\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty}\dfrac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$.

Thus, $\sum\limits_{n=1}^{\infty}\dfrac{\sin 4n}{4^n}$ is absolutely convergent.

20. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(-2)^n}{n^n}\right|} = \lim\limits_{n\to\infty}\dfrac{2}{n} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

21. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{n^2+1}{2n^2+1} = \lim\limits_{n\to\infty}\dfrac{1+1/n^2}{2+1/n^2} = \dfrac{1}{2} < 1$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

4. $b_n = \dfrac{n}{n^2+2} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since $\left(\dfrac{x}{x^2+2}\right)' = \dfrac{(x^2+2)(1)-x(2x)}{(x^2+2)^2} = \dfrac{2-x^2}{(x^2+2)^2} < 0$ for $x \geq \sqrt{2}$. Also, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{n}{n^2+2} = \lim\limits_{n\to\infty}\dfrac{1/n}{1+2/n^2} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n}{n^2+2}$ converges by the Alternating Series Test.

8. $\sum\limits_{k=1}^{\infty}\dfrac{2^k k!}{(k+2)!} = \sum\limits_{k=1}^{\infty}\dfrac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$\lim\limits_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\dfrac{2^{k+1}}{(k+2)(k+3)}\cdot\dfrac{(k+1)(k+2)}{2^k}\right| = \lim\limits_{k\to\infty}\left(2\cdot\dfrac{k+1}{k+3}\right) = 2 > 1$, so the series diverges.

Or: Use the Test for Divergence.

14. $\left|\dfrac{\sin 2n}{1+2^n}\right| \leq \dfrac{1}{1+2^n} < \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, so the series $\sum\limits_{n=1}^{\infty}\left|\dfrac{\sin 2n}{1+2^n}\right|$ converges by comparison with the geometric series $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2}\right)^n$ with $|r| = \frac{1}{2} < 1$. Thus, the series $\sum\limits_{n=1}^{\infty}\dfrac{\sin 2n}{1+2^n}$ converges absolutely, implying convergence.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $[p = 2 > 1]$, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

4. If $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}} \cdot \dfrac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)x\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| = \lim\limits_{n\to\infty} \sqrt[3]{\dfrac{1}{1+1/n}}\,|x| = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n}}$ diverges since it is a p-series $\left(p = \frac{1}{3} \le 1\right)$. Thus, the interval of convergence is $(-1, 1]$.

12. If $a_n = \dfrac{x^n}{n3^n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{3}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{|x|}{3}\left(\frac{1}{1+1/n}\right) = \frac{1}{3}|x|$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n3^n}$ converges when $\frac{1}{3}|x| < 1 \quad\Leftrightarrow\quad |x| < 3$, so $R = 3$. When $x = 3$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is the divergent harmonic series. When $x = -3$, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n}$ is the convergent alternating harmonic series (multiplied by -1). Thus, the interval of convergence is $[-3, 3)$.

18. If $a_n = \dfrac{n}{4^n}(x+1)^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)(x+1)^{n+1}}{4^{n+1}} \cdot \dfrac{4^n}{n(x+1)^n}\right| = \dfrac{|x+1|}{4} \lim\limits_{n\to\infty} \dfrac{n+1}{n} = \dfrac{|x+1|}{4}$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{n}{4^n}(x+1)^n$ converges when $\dfrac{|x+1|}{4} < 1 \quad\Leftrightarrow\quad |x+1| < 4 \quad [R = 4] \quad\Leftrightarrow$ $-4 < x+1 < 4 \quad\Leftrightarrow\quad -5 < x < 3$. When $x = -5$ or 3, both series $\sum\limits_{n=1}^{\infty} (\mp 1)^n n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty} |(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-5, 3)$.

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n (-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.