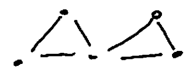
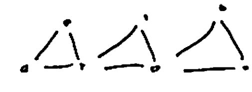
Practice #2 (solutions)

1) Define $G_1 =$  , $G_2 =$  , ...

then G_i has i cut-vertices and no bridges.

2) NO. If v is a cut-vertex in G, then $G - v$ is disconnected. By previous hw we know that

$$\overline{G-v} = \overline{G} - v$$

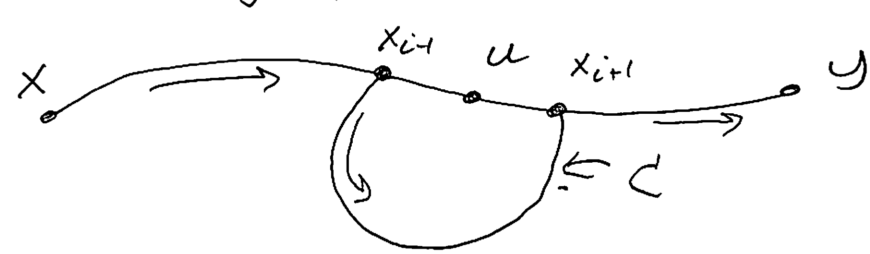
is cntd. Thus v is not a cut-vertex in $\overline{G}$.

3) $(\Rightarrow)$ If G is nonseparable, then by thm 7.3 any two edges lie on a common cycle. In particular any two adjacent edges lie on a common cycle.

$(\Leftarrow)$ Assume every pair of adj edges lies on a cycle. As the result vacuously holds for $n = 2$ we may assume $n \geq 3$. For a contradiction assume G has a cut-vertex u. Then by thm 6.2, $\exists$ vertices $x, y \ni$ every x-y path contains u. Let

$$P = (x = x_1, \ldots x_{i-1}, x_i = u, x_{i+1}, \ldots x_k = y)$$

be such a path. Then the edges $x_{i-1}x_i$ and $x_i x_{i+1}$ lie on a common cycle, C. Now we have



The arrows now indicate an x-y walk that does not involve u. $\Rightarrow\Leftarrow$.

$\therefore$ G is nonseparable.

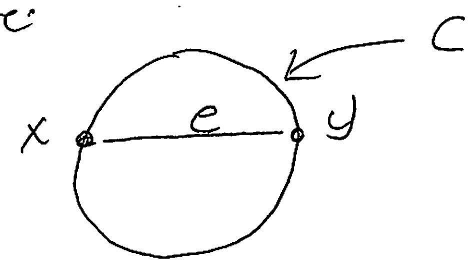
4) ($\Rightarrow$) If $\kappa(G-e) \geq 2$, then $G-e$ is nonseparable and any two vertices lie on a cycle. So x, y lie on a cycle.

($\Leftarrow$) Assume x, y lie on a common cycle in $G-e$. It will suffice to show that $\kappa(G-e) \neq 1$. (why?) For a contradiction assume it is and that u is a cut-vertex. Then

$$(G-e)-u = (G-u)-e$$

is a disconnected graph. As $G-u$ is cntd (why?) Then this means e is a bridge in $(G-u)$. Then e does not lie on any cycle in $G-u$. But if x, y lie on a cycle C in $G-e$ then in G we must have:

G:



If we delete u from G, then

a) u is not on C

b) u is on the bottom part of C

c) u is on the top part of C

either way e is on a cycle in $G-u$ $\Rightarrow \Leftarrow$.

5) Fix x, y so that $d(x,y) = \text{diam}(G)$. By Whitney's thm $\exists$ at least $k(G)$ int disj x-y paths

$$P_1, \dots P_{k(G)}$$

each w/ length $\geq \text{diam}(G)$. Note that

$$\text{length}(P_i) = |V(P_i) - x| \geq \text{diam}(G).$$

So $\text{length}(P_i) - 1 = |V(P_i) - x - y| \geq \text{diam}(G) - 1$

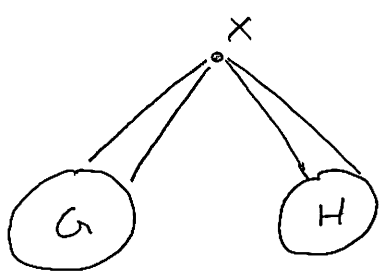
As the paths are int. disj. it follows that

$$n \geq k(G) \cdot (\text{diam}(G) - 1),$$

as the sets $V(P_i) - x - y$ are disjoint subsets of $V(G)$.

6) The picture is:

F:



It immediately follows that $\text{diam}(F) = 2$. By thm 9.1 we know $\chi(F) = \delta(F)$. If $u \in V(G)$, then

$$\deg_F u = \deg_G u + 1 \leq |V(G)|$$

and if $v \in V(H)$, then

$$\deg_F(v) = \deg_H v + 1 \leq |V(H)|.$$

As $\deg_F x = |V(H)| + |V(G)|$, it follows that

$$\delta(F) = \min(\delta(H), \delta(G)) + 1.$$

5) Let x, y be such that

$$d(x,y) = \text{diam}(G).$$

By whitney's thm $\exists$ at least $\kappa(G)$ int disj x,y paths. Each of these has length greater than (or equal to) $\text{diam}(G)$.