Solutions: #9

1) a- $G = K_2$

b- G can be either $C_3 = K_3$ or a star.

2) To show G has a perfect matching we will show that G satisfies Hall's condition. To this end let $X \subseteq U$ where

$$\{x_1 \ldots x_r\} = X$$

ordered $\ni$ $1 \le \deg(x_1) < \ldots < \deg(x_r)$. (we have strict inequalities as all deg in U have unique deg.)

It now follows that $\deg x_r \ge |X|$. Thus

$$|X| \le \deg x_r = |N(x_r)| \le |N(X)|$$

which shows G satisfies Hall's condition.

3) This is true. Observe:

G has a perfect matching iff $\alpha'(G) = n/2$ iff

$$\beta'(G) = n - n/2 = \alpha'(G)$$

where the last "iff" uses the fact that $\beta'(G) + \alpha'(G) = n$.

4) If $P = (e_1, e_2, \ldots, e_{2r+1})$ is an M-augmented path where $e_2, e_4, \ldots, e_{2r} \in M$ and $e_1, e_3, \ldots, e_{2r+1} \notin M$, then $M' = (M - \{e_2 \ldots, e_{2r}\}) \cup \{e_1, e_3, \ldots, e_{2r+1}\}$ is a set of indep. edges and $|M'| = |M| + 1$. So M' is a larger matching than M.

5) If G has a perfect matching M, then if player 1 picks the vertex x_i player 2 should pick <u>the</u> vertex x_{i+1} which is matched (under M) w/ x_i. Under this strategy, player 2 can always make a move. As the game cannot go on forever it follows that player 1 must loose, i.e., player 2 wins.

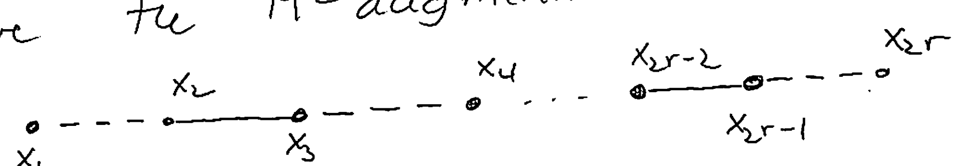
Now if G does not have a perfect matching, then Let M be a largest matching. The strategy player 1 should follow is:

① pick x_1 <u>not</u> covered in M.

② any time player 2 picks a vertex x_i covered by M, player 1 should pick the vertex x_{i+1} that is matched w/ x_i under M, i.e. $x_i x_{i+1} \in M$.

provided player 2 always chooses a vertex covered by M it follows that player 1 will win. We now claim that player 2 must in fact always choose a vertex covered by M. First, x_2 must be covered otherwise $M \cup \{x_1, x_2\}$ is a larger matching $\Rightarrow\Leftarrow$.

For another contradiction assume at some later pt player 2 picks a vertex x_{2r} that is not covered. Then we have the M-augmented path



which cannot exist by problem #4! Thus player 2 must always choose a vertex that is covered, guaranteeing player 1 a subsequent move.