Math 428
Graph Theory
Homework Set #9

Matchings & Edge Covers

1. Determine all graphs G, without isolated vertices, such that

   (a) $\beta'(G) = 1$

   (b) $\alpha'(G) = 1$ (Hint: Consider two cases: either G is acyclic or it is not.)

2. Let G be a connected bipartite graph with partite sets U and V. If $|U| = |V|$ and every vertex in U has a unique degree, then G contains a perfect matching.

3. Prove or disprove: A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

4. Given a matching M in G, an **M-alternating path** is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are not covered by M is an **M-augmented path**.

   Prove that if G has an M-augmented path, then G has a matching larger than M.

5. Two people play a game on a graph G, alternately choosing *distinct* vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

   Prove that the second player has a winning strategy if G has a perfect matching and otherwise the first player has a winning strategy.

   Hint: For the second part the first player should start with a vertex omitted by some maximum matching. Problem #4 might also be useful.