Math 428
Graph Theory
Homework Set #8

Whitney & Menger's Theorem

1. Assume G is nonseparable and let P be an $x - y$ path in G. Must there exist another $x - y$ path that is internally disjoint from P?

2. Recall the hypercube Q_n, i.e., the graph whose vertices are the set of all binary words of length n where two words are adjacent provided they differ in exactly one position. Earlier in the course we showed that $\delta(Q_n) = n$. So

$$\kappa(Q_n) \leq \lambda(Q_n) \leq \delta(Q_n) = n.$$

   Use Whitney's Theorem to prove that $\kappa(Q_n) \geq n$. Conclude that

$$\kappa(Q_n) = \lambda(Q_n) = \delta(Q_n) = n.$$

3. In class we talked about the following theorem:

   Theorem (9.5)**.** *Let $\kappa(G) \geq \ell$ and $x_1, \ldots, x_\ell$ be distinct vertices in G. Then $x_1, \ldots, x_\ell$ lie on a common cycle.*

   To show the converse of Theorem 9.5 is false, find, for each $\ell \geq 3$, a graph G_ℓ so that $\ell > \kappa(G_\ell)$ and any set of ℓ vertices in G_ℓ lie on a common cycle.

4. Let G be any graph such that $\kappa(G) \geq 5$. Fix distinct vertices u, v, and w. Prove that there exists two cycles C and C' that only have u and v in common and do not contain w.

5. Let G be any graph with $\kappa(G) \geq \ell$. Let U and W be disjoint subsets of $V(G)$ with $|U| = \ell = |W|$. Prove that there exists ℓ disjoint paths connecting U and W.

Optional Problems

1. Fix $\ell \geq 2$. Show that $\kappa(G) \geq \ell$ if and only if for each set S of ℓ distinct vertices and each choice of distinct $x, y \in S$ there is a cycle in G containing x and y but no vertex from $S - \{x, y\}$.