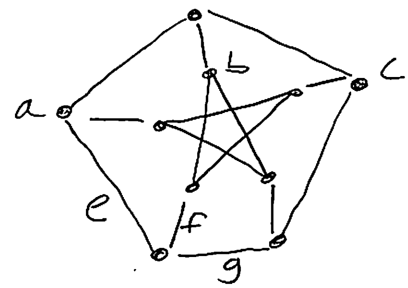
Hw 7: solutions

1) The Peterson graph is



A smallest vertex-cut is $\mathcal{U} = \{a, b, c\}$

A smallest edge-cut is $\Sigma = \{e, f, g\}$

$\therefore \quad k(PG) = 3 = \lambda(PG).$

Note: As PG is 3-reg, we must have $k = \lambda$ by thm 8.3.

2) We know from thm 8.2 that

$$\cancel{k} \le \lambda \le \underset{\text{kappa}}{k} \le \delta$$

so $\cancel{k} \le \delta$. We also know by HSL that

$$2m = \sum_{v \in V(G)} \deg v \ge n \cdot \delta \ge n \cdot k.$$

$$\therefore \quad m \ge \frac{nk}{2}.$$

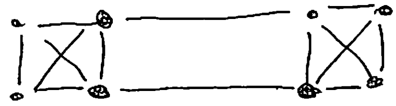
3) False! Consider

$$G = K_n + \underbrace{K_1}_{\text{call this } v}$$

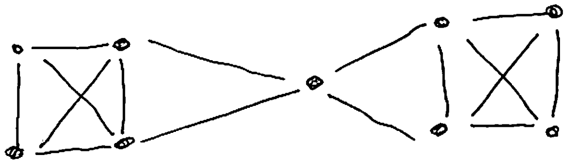
Then $k(G) = 0$, yet $k(\underbrace{G - v}_{= K_n}) = n-1.$

4)

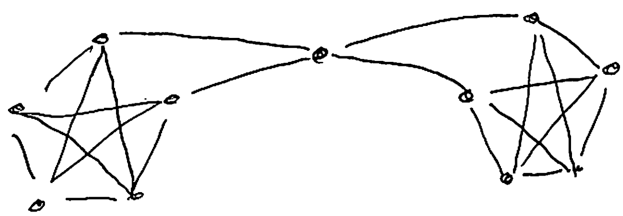
a -



b -



c -



$k = 1, \quad \lambda = 2, \quad \delta = 4 = \Delta.$

5) Let x, y be arbitrary vertices s.t. $x \not\sim y$. Recall that for any two sets A, B we have:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

If we let $A = N(x)$, $B = N(y)$, then

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)|$$

$$\geq \delta + \delta - (n-2) = 2\delta - n + 2$$

$$\geq k - 2 + 2 = k$$

as $\delta \geq (n+k-2)/2$.

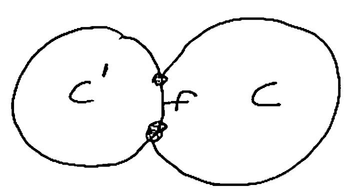
As $x \not\sim y$ then $N(x) \cup N(y) \subseteq V(G) - \{x, y\}$.

To show that $k(G) \geq k$, it will suffice to show that removing $\ell < k$ vertices from G does not disconnect G. Note $\ell \leq n-2$. So if we remove ℓ vertices then the resulting graph G' has at least two vertices x, y. As x, y have at least $k > \ell$ ~~vertices in~~ common n'bors in G, they <u>must</u> have at least one common n'bor in the smaller graph. $\therefore$ G' is still cntd.

6) To show $\lambda(G) \geq 3$, it will suffice to show that if $e, f \in E(G)$ then $G - \{e,f\}$ is still cntd. As e is on a cycle (in fact two of them) then e cannot be a bridge. So $G-e$ is cntd. Note that

$(G-e)-f$ is cntd iff f is not a bridge in $G-e$.

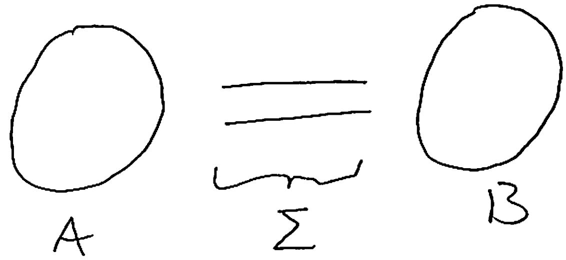
To show f is not a bridge in $G-e$, we must show it lies on a cycle in $G-e$. By assomption, in G, we know that f lies on two cycles:



where f is the only edge common to c' + c. Therefore e may lie on c' or c but not both. If e is on c' (wlog), then in $G-e$ f is still on the cycle c. Thus f is not a bridge in $G-e$ $\Rightarrow$ $G-e-f$ is cntd.

$\therefore$ $\lambda(G) \geq 3$.

7) As we know $\operatorname{diam}(G) = 2$, then by thm 9.1 we know $\lambda(G) = \delta(G)$. Further, if Σ is a min edge-cut, then we have

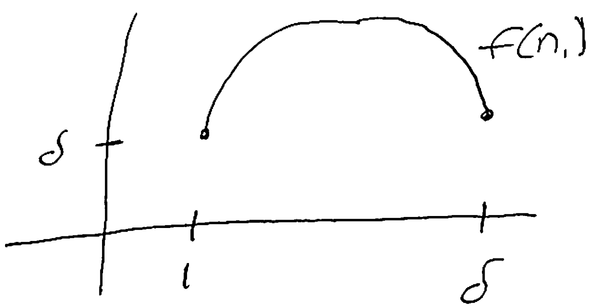


where $n_1 = |V(A)|$. From the pf of thm 9.1 we know

$$\lambda(G) \geq n_1(\delta - n_1 + 1) \geq \delta$$

and, as $\lambda = \delta$, then $f(n_1) = n_1(\delta - n_1 + 1) = \delta$.

Recall the graph of $f(n)$ is:



so in order for $f(n_1) = \delta$, then $n_1 = 1$ or δ.

Case 1: $n_1 = 1$

Then $A = K_1$ ✓

Case 2: $n_1 = \delta$.

Recall from the pf of thm 9.1, that each vertex in A is incident to at least one edge in Σ. As $|V(A)| = \delta$ & $|\Sigma| = \delta$, then each vertex in A is adj to exactly 1 edge in Σ. Now for $a \in V(A)$,

$$\delta \overset{*}{\geq} 1 + \deg_A a \geq \deg_G(a) \geq \delta$$

so $\deg_A a = \delta - 1 \Rightarrow A \cong K_\delta$.

(This ineq. follows as A has δ vertices, so $\deg_A a \leq \delta - 1$.)