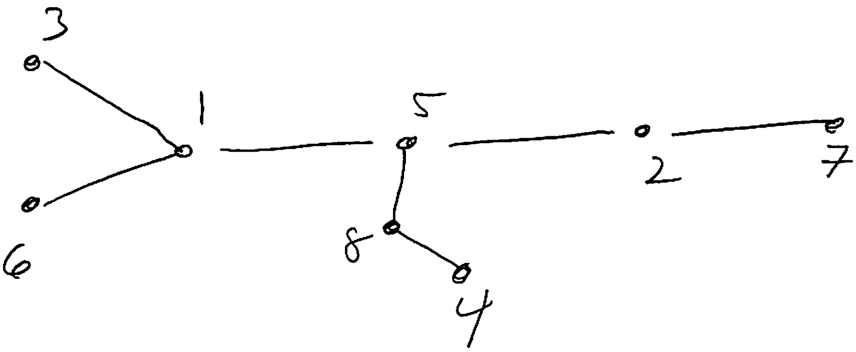
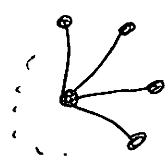
Solutions: HW5

1) a -



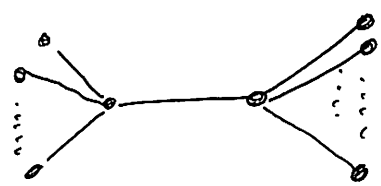
b - 11, 9, 6, 3, 7, 1, 11, 5, 13, 9, 1

2) a - Trees of the form:



(These trees are called stars.)

b -



c - Paths, P_n.

3) a - $\binom{n}{2}$ $(n-2)!$

choose the two leaves

of the remaining #'s, they must each appear in the prüfer code once. So this is just the # of ways to permute n-2 objects.

b - $\binom{n}{2}$ $\underbrace{(2^{n-2}-2)}$

choose two letters a + b.

For each of the 2 letters a + b. in our code we can choose which appears 1st, 2nd, 3rd, ...
The "minus 2" is b/c this approach ~~counts~~ includes the constant words

a .... a and b .... b

4) Induct on the order of the tree T.

Base case: $n=2$
In this case, the prufer code is $\emptyset$ and each vertex has deg 1. ✓

Now assume the result for all trees (labeled) w/ order n. Let T be a labeled tree of order n+1. Let ℓ be the smallest leaf and assume it is adj to a. Now let

$$w = w_1 \cdots w_{n-1}$$

be the prüfer code for T, then $w_1 = a$ and

$$w_2 \cdots w_{n-1}$$

is the code for $T - \ell$.

By induction, each label k in $T-\ell$ appears in

$$w_2 \cdots w_{n-1}$$

$(\deg_{T-\ell} k)-1$ times. To check that the result holds for the tree T and the code w it will suffice to consider only the labels a and ℓ. Certainly ℓ does not appear in w', so it does not appear in w. This is consistent w/ the fact that

$$\deg_T \ell = 1.$$

Now by induction a appears in w' $(\deg_{T-\ell} a)-1$ times. Since $1+\deg_{T-\ell} a = \deg_T a$ and

$$w = a \cdot w_2 \cdots w_{n-1}$$

we see that a appears in w

$$1+(\deg_{T-\ell} a)-1 = \deg_T a - 1$$

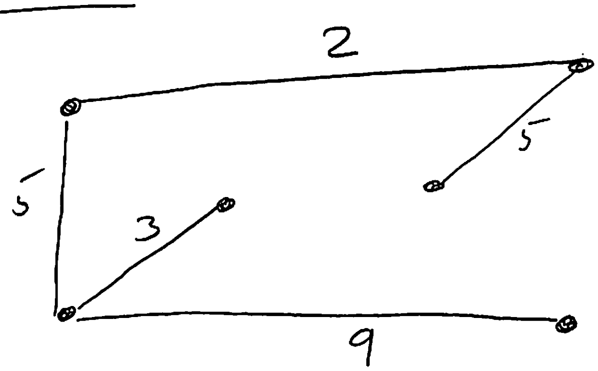
times as needed.

5) a- trees!

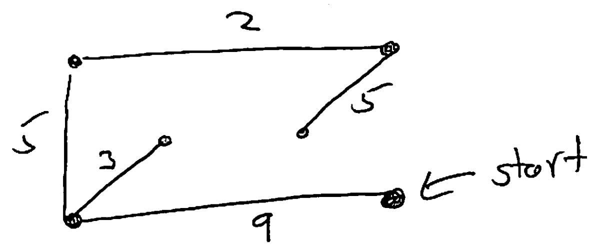
b- C_k, i.e. cycles w/ k edges since each spanning tree is determined by deleting 1 edge.

c- We know from a) that a graph has 1 spanning tree iff it is a tree. For a graph to have 2 spanning trees then it must have a cycle. But cycles have at least 3 edges. Now as, in b, this gives rise to at least 3 distinct spanning trees.

6) Kruskal's



Prim's



(Note: The answer in general depends where you start.)

7) Let T be the tree constructed by Kruskal's algorithm. As in the proof of krus. alg order its edges

$e_1, e_2, \ldots$

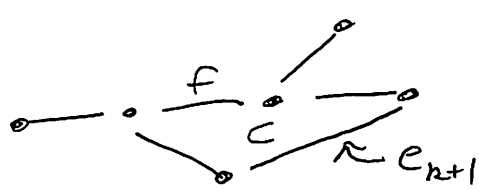
so that the edge e_i is choosen in the i^{th} step.

Now let Q be another min. spanning tree and assume the edges

$e_1, \ldots, e_n \in E(Q)$

but $e_{n+1} \notin E(Q)$. Just like in the pf. Note that $Q + e_{n+1}$ must have a cycle C

$Q + e_{n+1}$:



Now let f be an edge on C ∋. $f \notin E(T)$.

The exact same argument as that given in class shows $w(f) = w(e_{k+1})$. Since all edges have distinct weights then $f = e_{k+1}$ which is a contradiction $f \in E(Q)$ but $f = e_{k+1} \notin E(Q)$.

$\therefore$ $T = Q$.