Solutions: HW 4

1) Consider the graph

order $= 4$, size $= 3$.

2) Let F be a forest w/ components $T_1 \cdots T_k$ where T_i is a tree w/ order n_i + size $n_i - 1$.

so

$$|E(F)| = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k} n_i - 1 = \left(\sum_{i=1}^{k} n_i\right) - k$$

$$= |V(F)| - k.$$

3) Let G have order n. As $G-u$ is a tree:

$$|E(G)| - \deg u = |E(G-u)| = |V(G-u)| - 1$$
$$= n-1-1 = n-2.$$

we also know $G-v$ is a tree:

$$|E(G)| - \deg v = |E(G-v)| = |V(G-v)| - 1 = n-2.$$

$\therefore$ $\deg u = \deg v$

4) Let P & Q be longest paths in G.

$$P = (u_0 \dots u_k)$$

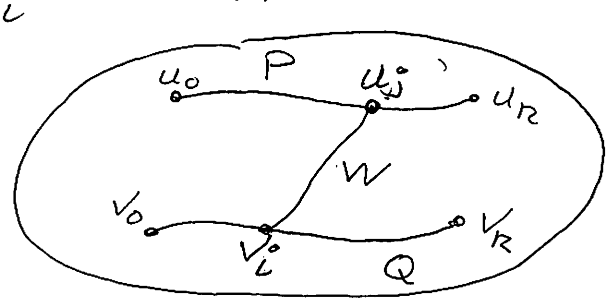
$$Q = (v_0 \dots v_k)$$

Assume for a contradiction that P & Q have <u>NO</u> vertices in common. As G is cntd we must have a $v_i - u_j$ path:

$$R = (w_0 \dots w_m)$$

where $w_0 = v_i$ & $w_m = u_j$. so

G:



Moreover we can choose W so only its end pts have vertices in common w/ P & Q. If $j \geq k/2$ & $i \geq k/2$, then the path

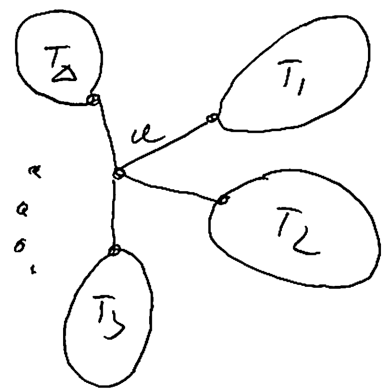
$$\underbrace{u_0 \dots u_j}_{\geq k/2} \cdot \underbrace{w_1 \dots w_{m-1}}_{\geq 1} \underbrace{v_{i-1} \dots v_0}_{\geq k/2}$$

has length $\geq k+1$ $\Rightarrow \Leftarrow$. The other cases are similar, and lead to paths $\geq k+1$.

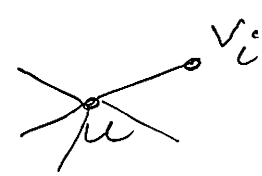
$\therefore$ P & Q must have some vertex in common.

5) Let u be a vertex in T w/ deg Δ.
So T looks like:

T:



So $T_1 \cdots T_\Delta$ are the cntd components of $T-u$. They must be trees. Either T_i is trivial, in which case T_i is just a single vertex v_i and then

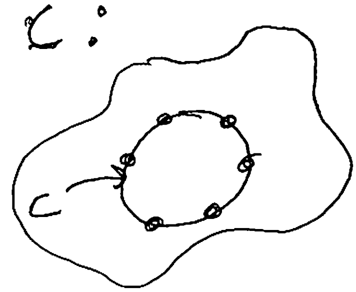


v_i is a leaf in T. Otherwise T_i is not trivial and by thm T_i has at least 2 leafs: l_1 & l_2. It follows that in T either l_1 or l_2 is still a leaf. (Possibly one of them is adj. to u in T, in which case it is no longer a leaf in the big tree.)
We conclude T has at least Δ leaves.

6) a) Assume G does not have a cycle. Then it is a forest, in which case all comp. are trees. If one of its trees is trivial then its a vertex of deg $0 \Rightarrow\Leftarrow$. Otherwise, if one of its trees is non-trivial then it must have a leaf (in fact two of them!) $\Rightarrow\Leftarrow$.

$\therefore$ G has a cycle.

b) If G has order n, then $G = C_n$. To see why, we know from a) that G has a cycle C:

G: 

No vertex on C can be adj to a vertex not on C. As G is cnt'd this implies the only vertices are those on C. Moreover if we had edges other than those on C then some vertex would have deg $\geq 3 \Rightarrow\Leftarrow$.

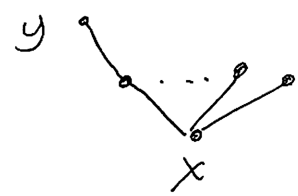
$\therefore$ $G = C_n$.

7) Let P be a path of max length in G. As $\text{diam}(G) = 2$, then

$$P = (u, x, v).$$

Now let $N_G(x)$ be the set of all n'bors of x in G. We claim $|N_G(x)| = n-1$, where $|V(G)| = n$. If $N(x) \neq V(G) \setminus \{x\}$ then as G is cntd $\exists$ some vertex y $\ni$ y is adj to a n'bor of x but $x \not\sim y$.

Then

G: 

which results in a path of length 3 $\Rightarrow \Leftarrow$.

Thus $N(x)$ consists of all the vertices in G except x. Thus x is isolated in $\overline{G}$.

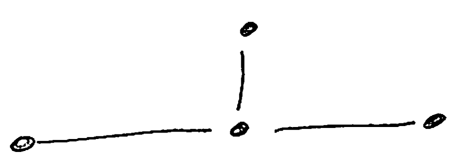
8) Let T be a tree of order n. We know
$$m := |E(T)| = n-1.$$

As $\bar{T}$ is the complement of T we must have
$$|E(\bar{T})| = \binom{n}{2} - |E(T)|$$
$$= \frac{n(n-1)}{2} - (n-1)$$

If $\bar{T}$ is a tree, then, as $n = |V(T)| = |V(\bar{T})|$,
$$\frac{n(n-1)}{2} - (n-1) = |E(\bar{T})| = n-1$$

so $n(n-1) = 4(n-1) \Rightarrow n^2 - 5n + 4 = 0$
$(n-4)(n+1) = 0$.
$\therefore$ $n = 4$ $(n \geq 1)$.

By checking all trees of order 4. we see P_4 is the only one $\rightarrow$ $\bar{P_4}$ is also a tree (In fact $\bar{P_4} = P_4$!).
The only other order 4 tree is:



whose comp. is NOT a tree.

8) As T is a tree, then

$$|E(T)| = |V(T)| - 1$$