Math 428
Graph Theory
Homework Set #4

Connected Graphs & Trees

1. Show that a graph of order n and size $n-1$ need not be a tree.
2. Show that any forest F of order n with k components has size $n-k$.
3. Let G be a connected graph. Assume G contains two vertices u, v such that $G-u$ and $G-v$ are trees. Show that $\deg u = \deg v$.
4. Prove that if P and Q are two longest paths in a connected graph (not necessarily a tree), then P and Q must have at least one vertex in common.
5. Prove that every tree T with maximum degree Δ, has at least Δ leaves.
6. (a) Prove that if every vertex in G has degree at least 2, then G must contain a cycle.
   (b) If G is connected and 2-regular, then what can we say about G?
7. (True or False) Let G be a (connected) graph with diameter 2. Show that $\overline{G}$ has an isolated vertex, i.e., a vertex with degree 0.
8. Find all trees T such that $\overline{T}$ is also a tree.

Optional Problems

1. Prove that a graph is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.
2. Let u be a vertex in a connected graph. Prove that it is possible to select shortest paths from u to every other vertex in G such that the union of the paths is a tree.